Probability Space

Stocks and returns

	AAA	BBB
2020	8%	3%
2021	-2%	6%
2022	12%	3%

This is an example of a probability space $\omega \in \Omega$

Events are subsets $E \subset \Omega$

Example:

$$E = \{AAA2020, BBB2021\}, F = \{AAA2020, AAA2022\}$$

Unions and Intersections of Events

Union: $E \cup F = \{AAA2020, BBB2021, AAA2022\}$

Intersection: $E \cap F = \{AAA2020\}$

Probability Measures

A probability measure is a function defined on events $\mu(E)$

- $\mu(E) \geq 0$
- $\mu(\Omega) = 1$
- if $E \cap F = \varnothing$ then $\mu(E \cup F) = \mu(E) + \mu(F)$

"the probability of disjoint events is the sum of the probabilities of the events"

Example: each is equally likely (1/6)

then: $\mu(\{AAA\}) = 1/2$

Mistakes

The head of a major tech corporation made a racist post on social media.

Which is more likely

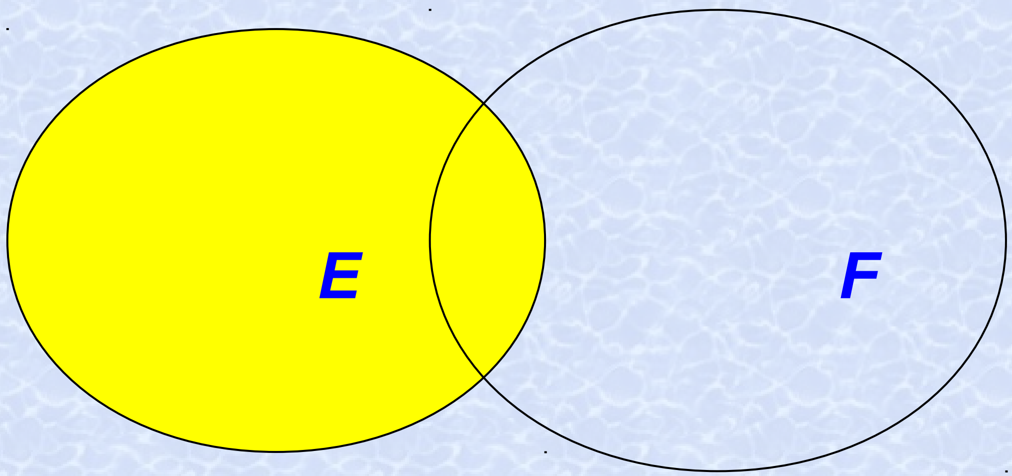
1. he is an engineer

2. he is an engineer and a white supremacist

Conditional Probability

$\mu(E|F) = \mu(E \cap F)/\mu(F)$

what does F tell you about E? information or learning



$\mu(-2\%) = 1/6, \mu(-2\%|AAA) = 1/3$

Random Variables

A random variable $x{:}\Omega \to \Re$ assigns each point in the probability space a real number

For example: the random variable is the return

notice that it makes perfectly good sense to add, subtract, multiply, etc. random variables

probabilities of random variables are computed from the underlying probability space

Example

x is the return

$x(AAA2020) = 8$

$\Pr(x = 3) = 1/3$

$\Pr(2x = 16) = 1/6$

Expectation

The expectation of a random variable is the probability weighted average

$$Ex = \sum_{\omega \in \Omega} x(\omega)\mu(\omega)$$

Example

$$Ex = (1/6)8 + (1/6)(-2) + (1/6)12 + (1/3)3 + (1/6)6$$
$$= 5$$

Some Important Facts

$$E(x+y) = Ex + Ey$$

if *a* is a constant

$$Eax = aEx$$

but if *y* is a random variable

$Eyx \neq EyEx$ in general

Variance

The variance of a random variable is

$$\mathrm{var}x = E(x - Ex)^2 = Ex^2 - (Ex)^2$$

Expectation measures central tendency

variance measures uncertainty or risk

Example

$$\mathrm{var}x = (1/6)3^2 + (1/6)7^2 + (1/6)7^2 + (1/3)2^2 + (1/6)1^2$$
$$= 19\,1/3$$

Standard Deviation

is the square-root of the variance

measured in conformable units to the mean

it makes sense to talk about “within two standard deviations of the mean”

Example:

$$\mathrm{var} x = 19\,1/3, \mathrm{sd} x = 3.06$$

rule of thumb: 95% of the time within two standard deviations of the mean

so return can be thought of as 5% plus or minus 6%

Conditional Expectation

$$E(x|AAA) = 9, E(x|BBB) = 4$$

so AAA give a substantially higher expected return than BBB

$$\mathrm{var}(x|AAA) = 131/3, \mathrm{var}(x|BBB) = 6/3$$

so BBB is much less risky

Large Independent Samples

repeatedly draw from the same population

for example: polling over voting, other surveys

(with or without replacement? In a large sample it doesn’t really matter)

independent draws or nearly so

Central limit theorem: the average follows approximately a “normal” distribution

In particular 95% of the time the average will be within two standard deviations of the true mean

Binomial Sampling

each voter has an independent probability p of voting for Labour

we poll a number N voters, record the random variable x equal to 1 if they say they are voting for Labour and 0 if the say they are not

mean and variance of x

$$Ex = (1-p)0 + p1 = p$$

$$\mathrm{var}x = (1-p)(0-p)^2 + p(1-p)^2 = p(1-p)$$

Sample Average

sample average: $\overline{x} = (1/N)\sum_{i=1}^{N} x_i$

mean: $E\overline{x} = (1/N)NEx_i = p$

variance: $\mathrm{var}\overline{x} = (1/N)^2 N \mathrm{var} x_i = p(1-p)/N$

standard deviation $\mathrm{sd}\overline{x} = \sqrt{p(1-p)}/\sqrt{N}$

does not depend on population size

Diversification

many risky stocks with independent returns

expected return $\overline{x}$ and variance σ^2

put 100 dollars in one stock

expected return $100x$ variance $10000\sigma^2$

put 1 dollar in each stock

expected return $100x$ variance $100\sigma^2$

diversification reduces risk

for example: investing in a index mutual fund that holds many stocks rather than in an individual stock entails less risk

What if Returns Are Not Independent?

finance experts compute β measuring how correlated the stock is with the market

adding a stock with $\beta > 1$ increases risk, $\beta < 1$ reduces it

Concepts

- **probability**, **conditional probability**
- **random variable**, **expectation**, **variance**
- **risk**, **diversification**

Skill

Given information about stock returns

Find the expected return and variance