Lotteries, Sunspots and Incentive Constraints

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Introduction

- How to model idiosyncratic risk?
- Standard answers: Incomplete markets; participation constraints
- Why does Bill Gates have such an undiversified portfolio?
- Need to introduce moral hazard
- This was done many years ago by Prescott and Townsend in “lottery economies”
- Lotteries widely used in applied work with indivisibilities (Hansen, Rogerson, others)
- Still controversial and not widely used in the analysis of asset markets
- Recent work by Bennardo, Bennardo & Chiappori
- Connection between sunspots and lotteries in indivisibility case: Shell, Wright, Garrett and others

Goals

- A model in which rich face incentive constraints and poor face participation constraints
- Biased portfolios of rich individuals cannot be explained by incomplete markets or liquidity constraints
- Lack of insurance for workers against market conditions cannot be explained by moral hazard or adverse selection
- General equilibrium framework in which there are identifiably different classes of households, but within a class, there is private information
- Evaluate problem from perspective of demand (response of demand to prices) in order to incorporate individual as one of many identifiable types in a GE model

A Motivating Example

continuum of traders ex ante identical

two goods $j = 1,2$; c_j consumption of good *j*

utility is given by $\tilde{u}_1(c_1) + \tilde{u}_2(c_2)$

each household has an independent 50% chance of being in one of two states, $s = 1,2$

endowment of good 1 is state dependent
$\omega_1(2) > \omega_1(1)$; endowment of good 2 fixed at ω_2.

In the aggregate: after state is realized half of the population has high endowment half low endowment

Gains to Trade

after state is realized
low endowment types purchase good 1 and sell good 2

before state is realized
traders wish to purchase insurance against bad state

unique first best allocation
all traders consume $(\omega_1(1)+\omega_1(2))/2$ of good 1, and ω_2 of good 2.

Private Information

idiosyncratic realization private information known only to the household

first best solution is not incentive compatible

low endowment types receive payment
$(\omega_1(2)-\omega_1(1))/2$
high endowment types make payment of same amount

high endowment types misrepresent type to receive rather than make payment

Incomplete Markets

prohibit trading insurance contracts

consider only trading ex post after state realized

resulting competitive equilibrium

- equalization of marginal rates of substitution between the two goods for the two types
- low endowment type less utility than the high endowment type

Mechanism Design

purchase $x_1(1) > 0$ in exchange for $x_1(2) < 0$

no trader allowed to buy a contract that would later lead him to misrepresent his state

assume endowment may be revealed voluntarily, so low endowment may not imitate high endowment

incentive constraint for high endowment

$$\tilde{u}_1\,(\omega_1(2) + x_1(2)) + \tilde{u}_2\,(\omega_2 + x_2(2))$$
$$\geq \tilde{u}_1\,(\omega_1(2) + x_1(1)) + \tilde{u}_2\,(\omega_2 + x_2(1))$$

- Pareto improvement over incomplete market equilibrium possible since high endowment strictly satisfies this constraint at IM equilibrium
- Need to monitor transactions

Lotteries and Incentive Constraints

one approach: X space of triples of net trades satisfying incentive constraint
use this as consumption set

our approach: enrich the commodity space by allowing sunspot contracts (or lotteries)

1) X may fail to be convex

2) incentive constraints can be weakened - they need only hold on average

$$E|_2\, \tilde{u}_1(\omega_1(2) + x_1(2)) + \tilde{u}_2(\omega_2 + x_2(2))$$
$$\geq E|_1\, \tilde{u}_1(\omega_1(2) + x_1(1)) + \tilde{u}_2(\omega_2 + x_2(1))$$

The Base Economy

households of *I* types $i = 1, \ldots, I$
individual household denoted by $h \in H^i = [0,1]$
J traded goods $j = 1, \ldots, J$

random "sunspot" variable σ uniformly distributed on [0,1]

idiosyncratic risk
household of type *i* consumes in finite number of states $s \in S^i$ where probability is $\pi^i(s)$ satisfying $\sum_{s \in S^i} \pi^i(s) = 1$
states are drawn independently by households

contracts for delivery contingent on sunspot and the individual state of the household
(note that not on state of other households - simplifies notation)

$x_j^i(s,\sigma,h) \in \Re$ net amount of good *j* delivered to household *h* of type *i* when the idiosyncratic state is *s* and the sunspot state is σ.

1) trading
2) states and sunspots realized
3) deliveries
4) consumption

type *i* and household *h*, sunspot σ
net trading plan $x^i(\sigma,h) \in \Re^{JS^i}$
must belong to *feasible trading* set X^i
endowments incorporated directly into feasible net trade set

net trades are observable, consumption may not be

utility $u^i : X^i \to \Re$

for each household *h* of type *i* sunspots induce a probability measure μ^i over X^i
a *lottery* for type *i*

utility of lottery μ^i is

$$u^i(\mu^i) = \int_{X^i} u^i(x^i)d\mu^i(x^i)$$

Incentive Constraints

constraints that must hold "on average"

feasible reports $F_s^i \subseteq S^i$ represent the reports that a trader can make about his state when his true state is *s* without being contradicted by either public information or physical evidence.

Feasible Truthtelling: For all $s \in S^i$, $s \in F_s^i$.

Feasible Misrepresentation: If $s' \in F_s^i$, then $X_{s'}^i \subseteq X_s^i$.

$x^i(\sigma, h)$ is called *incentive compatible* if for all $s' \in F^i(s)$

$$\int u_s^i(x_s^i(\sigma,h))d\sigma - \int u_s^i(x_{s'}^i(\sigma,h))d\sigma \geq 0$$

Perfect Competition with Sunspots

sunspot equilibrium with transfer payments
socially feasible sunspot allocation χ
non-zero measurable price function $p(\sigma) \in \Re_+^J$; price of delivery contingent on an idiosyncratic state is $\pi^i(s)p(\sigma)$
all types *i* and almost all $h \in [0,1]$; $\chi^i(h,\cdot)$ maximizes $u^i(\tilde{\chi}^i)$ over individual sunspot allocations $\tilde{\chi}^i$ satisfying

sunspot budget constraint

$$\int \sum_{s \in S^i} \pi^i(s)p(\sigma)x^i(\sigma)[s]d\sigma \le \int \sum_{s \in S^i} \pi^i(s)p(\sigma)\chi^i(h,\sigma)[s]d\sigma$$

and incentive constraints $\int u_s^i(x_s^i(\sigma,h))d\sigma - \int u_s^i(x_{s'}^i(\sigma,h))d\sigma \ge 0$.
transfer payments depend only on types

$$\int \sum_{s \in S^i} \pi^i(s)p(\sigma)\chi^i(h,\sigma)[s]d\sigma = \int \sum_{s \in S^i} \pi^i(s)p(\sigma)\chi^i(\hat{h},\sigma)[s]d\sigma \text{ a.e.}$$

Theorem 3.1.2 First Welfare Theorem Every sunspot equilibrium allocation is Pareto efficient.

Theorem 3.1.3 Second Welfare Theorem For every Pareto efficient allocation with equal utility there are prices forming a sunspot equilibrium.

Theorem 3.1.4 Existence Theorem There is at least one sunspot equilibrium with endowments.

The Stand-in Consumer Economy

This is the one with 2.3 children and 1.8 automobiles

What can the average household purchase?

$Y^i \equiv \text{Closure}(\text{ConvexHull}\{y^i \in \Re^J | \exists x^i \in X^i, y^i = A^i x^i\})$.

$y^i \in Y^i$ may be allocated to households of type *i* by means of a lottery over the consumption set X^i

utility of average household get from a bundle $y^i \in Y^i$

bundle is allocated to individual households optimally, then

$$v^i(y^i) = \sup \int u^i(x^i) d\mu^i(x^i)$$

subject to support $\mu^i \subseteq X^i$, $\int \sum_{s \in S^i} \pi^i(s) x^i d\mu^i(x^i) \le y^i$,

$$\int u_s^i(x_s^i(\sigma,h)) d\sigma - \int u_s^i(x_{s'}^i(\sigma,h)) d\sigma \ge 0 .$$

an allocation *y* is a vector $y^i \in Y^i$ for each type

allocation socially feasible if $\sum_i y^i \leq 0$

stand-in consumer equilibrium with transfer payments

non-zero price vector $p \in \Re_+^M$
socially feasible allocation *y*

type *i* y^i should maximize $v^i(\tilde{y}^i)$ subject to $p \cdot \tilde{y}^i \leq p \cdot y^i$, $y^i \in Y^i$

a stand-in consumer allocation is *equivalent* to either a sunspot allocation if the allocations use the same aggregate resources and yield the same utility to each type.

The Role of Lotteries and Incentive Constraints

Back to the insurance example

Proposition 4.1.1 Suppose that $\tilde{u}_1$ exhibits declining absolute risk aversion, and that $\tilde{u}_2$ is strictly concave. If μ^i solves the stand-in consumer problem

$$v^1(y^1) = \max \int u^1(x^1) d\mu^1(x^1)$$

subject to $\text{support}\, \mu^1 \subseteq X^1, \int \sum_{s \in S^i} \pi^1(s) x^1 d\mu^1(x^1) \leq y^1, \int g^i(x^i) d\mu^i(x^i) \leq 0$,

then μ^1 is a point mass on a single point.

This generalizes

Infinite Horizon Case: Single Consumer

continuum of ex ante identical households consumes in periods
$t = 1,2,\ldots$.

risk idiosyncratic only
household is in one of finitely many individual states $\eta \in I$

individual states Markov
transition probabilities $\pi_{\eta_t \eta_{t-1}} > 0$

number of households moving between states deterministic

initial condition is steady state

Sunspots and Histories

each period *t*, after idiosyncratic states realized
sunspot (public randomization device, lottery) σ_t is drawn from i.i.d. uniform distribution

$s = (\eta_1, \sigma_1, \eta_2, \sigma_2 \ldots, \eta_t, \sigma_t)$ history of idiosyncratic states and sunspots

length of the history $t(s) = t$
τth state $\eta_\tau(s)$
histories ordered in natural way $\tilde{s} \geq s$
$s-1$ history that precedes *s*
$\eta_t(s)$ final state η_t

each time, given initial distribution of states, transition probabilities and sunspot process induce probability measure over length *t* histories $\pi(ds)$

Consumption Plans and Utility

J different consumption goods

allocation assigns household net-trade $x_s \in \Re^J$ contingent on idiosyncratic history of household

households have common discount factor $1 > \delta \geq 0$

x is a history contingent consumption plan

$$U(x) = (1-\delta)\sum_{t=1}^{\infty} \delta^{t-1} \int_{t(s)=t} u(x_s, \eta_s)\pi(ds)$$

u is concave and bounded below

Participation Constraints

$V(\eta_0) \le (1-\delta)\sum_{t=0}^{\infty} \delta^t \int_{t(s)=t} u(0,\eta_s)\pi(ds|\eta_0)$ defines individually rational utility levels

participation constraint

$$(1-\delta)\sum_{t=t(\tilde{s})}^{\infty} \delta^t \int_{t(s)=t} u(x_s,\eta_s)\pi(ds|\eta_{\tilde{s}}) \ge V(\eta_{\tilde{s}})$$

note $V(\eta) = -\infty$ is allowed

A *feasible report* is $F_\eta \subseteq I$

If misrepresentation is possible it will not be discovered at a later date t

Feasible reporting plan is map $\mu:(s,\eta) \to \tilde{\eta}$ such that $\tilde{\eta} \in F_\eta$

induces a map $S \xrightarrow{\mu} S$ from histories to histories
the history that is reported given the true history

induces a map $X \xrightarrow{\mu} X$ from allocations to allocations;
the net trade corresponding to the reported history

an allocation incentive compatible if
$U(x) \geq U(\mu(x))$ for all reporting plans μ

Timing

- Draw type
- Announce type
- Draw sunspot
- Contractual deliveries made
- Consume

Aggregate Excess Demand

Aggregate excess demand for all households

$$\int_{t(s)=t} x_s \pi(ds)$$

Direct Utility of the Stand-in Consumer

y_t is period *t* excess demand
allocate this amount efficiently among the *ex ante* identical group

$v(y) = \max_x U(x)$ subject to
x incentive compatible, *x* individually rational
$\int_{t(s)=t} x_s \pi(ds) \leq y_t$

Indirect Utility of the Stand-in Consumer

$v(p) = \max_y v(y)$ subject to
$\sum_{t=1}^{\infty} p_t y_t \leq 0$

Recursive Characterization of Indirect Utility

space *V* of $(v,w) \in \Re^I \times \Re$ of type contingent utility and wealth

$G(\eta_{T-1}, p^T)$ convex subset of *V* for infinite price vector $p^T = (p_T, p_{T+1}, \ldots)$

η_{T-1} is announcement at time $T-1$, v_{η_T} is the realized utility at time *T*

Let $\overline{V}(\eta_{T-1}, p^T, w)$ be the greatest achievable utility without incentive or participation constraints

characterize "equilibrium" *G*'s

- Boundedness

If $(v,w) \in G(\eta_{T-1}, p^T)$ then $\sum_{\eta} v_{\eta \in I} \pi(\eta | \eta_{T-1}) \le \overline{V}(\eta_{T-1}, p^T, w)$

Question: can $\overline{V}$ be infinite, yet w/ incentive and participation constraints utility is bounded?

Recursive Relations

Characterize $G(\eta_{T-1}, p^T)$ in terms of $G(\eta_T, p^{T+1})$ ("self-generation")

Suppose that $(v, w) \in G(\eta_{T-1}, p^T)$

Let η_T be the announcement
Must find

- consumption plan $x(\eta_T, \sigma)$
- new wealth $w(\eta_T, \sigma)$
- new utilities $v_{\eta_{T+1}}(\eta_T, \sigma)$, for all $\eta_{T+1} \in I$

Feasibility

- consumption plan $x(\eta_T,\sigma)$
- new wealth $w(\eta_T,\sigma)$
- new utilities $v_{\eta_{T+1}}(\eta_T,\sigma)$, for all $\eta_{T+1}\in I$

recursivity

$(v(\eta_T,\sigma),w(\eta_T,\sigma))\in G(\eta_T,p^{T+1})$ for all η_T,σ

budget feasibility

$$w=\sum_{\eta_T\in I}\int\left[\left(p_T x(\eta_T,\sigma)+w(\eta_T,\sigma)\right)\pi_{\eta_T\eta_{T-1}}\right]d\sigma$$

present value of utility

$$v_{\eta_T}=\sum_{\eta_T\in I}\int\left[(1-\delta)u\left(x(\eta_T,\sigma),\eta_T\right)+\delta\sum_{\eta_{T+1}\in I}v_{\eta_{T+1}}(\eta_T,\sigma)\pi_{\eta_{T+1}\eta_T}\right]\pi_{\eta_T\eta_{T-1}}d\sigma$$

Participation and Incentive Compatibility

- consumption plan $x(\eta_T,\sigma)$
- new wealth $w(\eta_T,\sigma)$
- new utilities $u_{\eta_{T+1}}(\eta_T,\sigma)$, for all $\eta_{T+1}\in I$

$$v_{\eta_T} \geq V(\eta_T)$$

if $(\eta,\tilde{\eta})\in F$ then

$$\int\left[(1-\delta)u(x(\eta_T,\sigma),\eta_T)+\delta\sum_{\eta_{T+1}\in I}u_{\eta_{T+1}}(\eta_T,\sigma)\pi_{\eta_{T+1}\eta_T}\right]d\sigma \geq$$
$$\int\left[(1-\delta)u(x(\tilde{\eta}_T,\sigma),\eta_T)+\delta\sum_{\eta_{T+1}\in I}u_{\eta_{T+1}}(\tilde{\eta}_T,\sigma)\pi_{\eta_{T+1}\eta_T}\right]d\sigma$$

Results

Lemma 1: *G* is convex

Lemma 2: generation operator is monotone

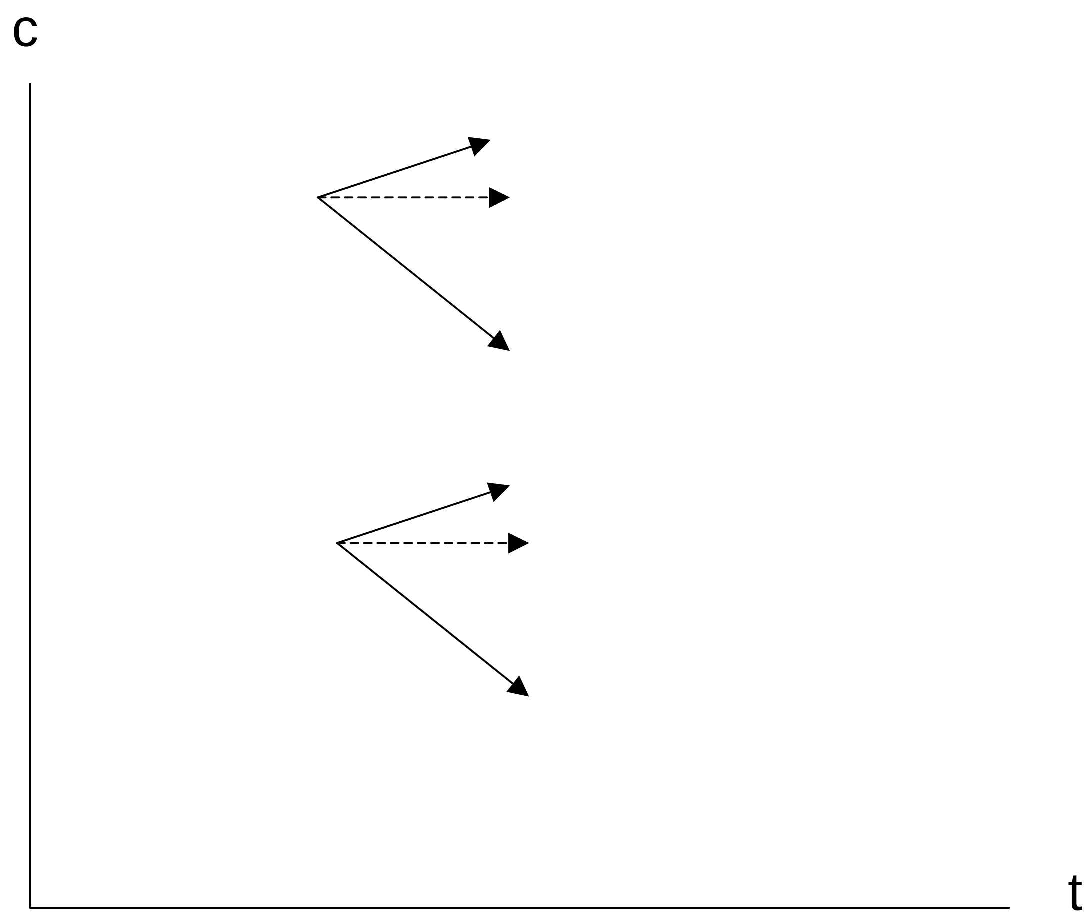
Obvious starting place for finding generation operator: start at solution without incentive and participation constraints, then work down to the fixed point

Two State One Good CES

With incentive constraints only: Atkeson and Lucas [1992]
Consumption is a logarithmic random walk with negative bias

With participation constraints only: there is a maximum and minimum level of consumption; a favorable state always gets the maximum. Each unfavorable realization leads to a drop in consumption until the minimum is reached

What happens with both constraints?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