A Large Deviation Theorem for Triangular Arrays

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1. Introduction

In examining repeated games with signals converging to diffusion processes, the equilibrium cutpoint that determines the probability of punishment may grow asymptotically relative to the standard error of the signal, as for example, in Fudenberg and Levine [2007]. The standard central limit theorem does not provide a useful estimate of the way that this probability converges to that of the corresponding normal distribution, but in some cases a large deviations version of the central limit theorem can be used. The most useful result that we have been able to find is that of Feller [1972], however this applies only to i.i.d. random variables, and not to triangular arrays. This note provides the additional uniformity assumptions needed to adapt the Feller proof to the case of triangular arrays and shows how to adapt the proof.

2. The Setup

As indicated, we extend an argument concerning i.i.d. random variables from Feller [1972, pp. 548-553] to the case of triangular arrays. We adopt Feller's notation to the maximum extent feasible. We suppose that we are given for each n a sequence $Z_i^n\ i = 1,\ldots,n$ of *i.i.d.* random variables with zero mean, variance σ_n^2 and distribution F_n. We define

$$z_n = \sum_{i=1}^{n} Z_i^n .$$

This has distribution F^{n*}, while the normalized sum $z_n / \sigma_n \sqrt{n}$ has distribution F^n.

Let Φ, ϕ respectively denote the c.d.f. and density of the standard normal distribution. Recall that the *cumulant generating function*[2] is defined as the logarithm of the generating function

$$\psi_n(\zeta) \equiv \log \int_{-\infty}^{\infty} e^{\zeta x} F_n(dx).$$

By the usual properties of the moment generating function, z_n has cumulant generating function $n\psi_n(\zeta)$. The derivatives of the cumulant generating function at zero are the corresponding central moments: $\psi_n{}'(0) = EZ_i^n, \psi_n{}''(0) = \mathrm{var}(Z_i^n)$ and so forth. Our goal is to prove the following result:

[2] Also called the bilateral Laplace transform.

Large Deviations Theorem: *Suppose*

1. *For some* $\overline{s} > 0$ *and all* $0 \leq \zeta \leq \overline{s}$ *there is a continuous function* $\psi^2(\zeta) > 0$ *and constant* $B > 0$ *such that*

$$\lim_{n\to\infty} \sup_{0\leq\zeta\leq\overline{s}} \left|\psi_n ''(\zeta) - \psi^2(\zeta)\right| \to 0$$

and that

$$\sup_{0\leq\zeta\leq\overline{s}} \mid \psi_n '''(\zeta) \mid , \mid \psi_n ''''(\zeta)\zeta \mid\mid \psi_n '''''(\zeta)\zeta^2 \mid < B$$

2. $\sigma_n \to \sigma$, $M_{3n} \equiv E\left|Z_n^i\right|^3 \to M_3 < \infty$

3. $n^{-1/6}x_n \to 0$

4. $x_n \to \infty$

Then

$$\frac{1 - F^n(x_n)}{1 - \Phi(x_n)} \to 1$$

3. Basic Facts

Our proof will make use of some basic facts. The first is a more standard version of the central limit theorem, also from Feller.

Berry-Esseen Theorem:[3] *for all* x

$$\mid F^n(x) - \Phi(x) \mid \leq \frac{9E \mid Z_i^n \mid^3}{\sqrt{n}\sigma_n^3}.$$

We also use some basic results about the standard normal distribution.

Lemma 1:

1. $\lim_{x\to\infty} \dfrac{x^{-1}\phi(x)}{1 - \Phi(x)} = 1$

2. *there is a constant* K *such that if* $x > 0$, $1 - \Phi(x) \leq Kx^{-1}\phi(x)$

[3] Feller uses the constant 3 instead of 9; Wolfram gives 33/4 which is slightly smaller than 9.

proof:

1) Apply L'Hopital's rule

$$\lim_{x\to\infty} \frac{x^{-1}\phi(x)}{1-\Phi(x)} = \lim_{x\to\infty} \frac{-(1+x^{-2})\phi(x)}{-\phi(x)} = 1\,.$$

2) From property 1, there is an $\bar{x}$ such that for $x \geq \bar{x}$, $1-\Phi(x) \leq 2x^{-1}\phi(x)$. On the other hand, for $0 \leq x \leq \bar{x}$ we have

$$1-\Phi(x) \leq \left[\max_{0\leq x\leq \bar{x}} \frac{1-\Phi(x)}{x^{-1}\phi(x)}\right] x^{-1}\phi(x)$$

so that we may take

$$K = \max\{2, \left[\max_{0\leq x\leq \bar{x}} \frac{1-\Phi(x)}{x^{-1}\phi(x)}\right]\}.$$

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Lemma 2: *If Assumptions 1 and 2 of the Large Deviations Theorem hold then* $\psi^2(0) = \sigma^2$.

Proof: Because it is the cumulant generating function for Z_i^n, $\psi_n{}''(0) = \sigma_n^2$. By Assumption 2 $\sigma_n^2 \to \sigma$. By Assumption 1 if $\zeta \to 0$ then $\psi_n{}''(\zeta) \to \psi^2(0)$. But by a diagonalization argument we can then choose $\zeta \to 0$ sufficiently fast that $\psi_n(\zeta) \to \sigma^2$.

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4. The "Associated" Distribution

Feller's proof replaces the normalized sum $z_n / \sigma_n \sqrt{n}$ and its cdf F^{n*} with a different random variable. This "associated" random variable has probability measure given by the cdf V_s^{n*} where s is a positive constant and

$$V_s^{n*}(x) \equiv \int_0^x e^{-n\psi_n(s)} e^{sy} dF^{n*}(y)$$

Notice that this has a much thicker right tail than F^{n*}. The idea is that by applying a version of the CLT, the Berry-Esseen Theorem above, to V^{n*}, we can pull this back to the thinner tailed F^{n*} to get a bound that will apply even for large values of x.

First we develop some basic properties of V_s^{n*}

Lemma 3*:*

$$\int_{-\infty}^{\infty} dV_s^{n*}(x) = 1$$

Proof:

$$\int_{-\infty}^{\infty} dV_s^{n*}(x) \equiv \int_{-\infty}^{\infty} e^{-n\psi_n(s)} e^{sx} dF^{n*}(x) = e^{-n\psi_n(s)} e^{\log \int_{-\infty}^{\infty} e^{sx} dF^{n*}(x)}$$

and the result follows from the fact that z_n has cumulant generating function $n\psi_n$.

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Lemma 4: *V_s^{n*} has mean $n\psi'(s)$ and variance $n\psi''(s)$.*

Proof: Follows by computing the cumulant generating function for V^{n*}

$$\begin{aligned}\psi_{V_s^{n*}}(\zeta) &= \log \int_{-\infty}^{\infty} e^{\zeta x} dV_s^{n*}(x) = -n\psi_n(s) + \log \int_{-\infty}^{\infty} e^{(s+\zeta)x} dF^{n*}(x) \\ &= -n\psi_n(s) + n\psi_n(s+\zeta)\end{aligned}$$

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Lemma 5: *V_s^{n*} is the cumulative distribution function of the sum of i.i.d. random variables with distribution*

$$V_{ns}(x) \equiv \int_0^x e^{-\psi_n(s)} e^{sy} dF_n(y)$$

Proof: This follows from the basic properties of the exponential function: multiplying a density by an exponential of the integrand commutes with the taking of convolutions.[4]

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5. Sketch of the Proof

We want to give a sufficient condition for

$$\left| \frac{1 - F^n(x_n)}{1 - \Phi(x_n)} - 1 \right| \to 0 \text{ as } x_n \to \infty .$$

The idea is to introduce an intermediate quantity A_n and give a sufficient condition that

$$\left| \frac{1 - F^n(x_n)}{A_n} - 1 \right| \to 0$$

[4] Note the basic one-tailed nature of the argument: we can thicken the tail while preserving convolutions only if we multiply by an exponential. While this thickens one tail, it also thins the other tail.

and

$$\left| \frac{A_n}{1 - \Phi(x_n)} - 1 \right| \to 0 \, ,$$

the two together then giving the desired result. The first step will follow by applying a version of the CLT, the Berry-Esseen Theorem, which gives a precise rate of convergence, to the thick tailed V^{n*}. The second step shows that when we thicken the tail by multiplying by a carefully chosen exponential we do not shift $V_s^{n^*}$ too much to the right. To carry out this second step we need the key condition $n^{-1/6} x_n \to 0$.

6. Proof of the Main Theorem

6.1. First step

Invert the relationship $dV_s^{n*}(x) = e^{-n\psi_n(s)} e^{sx} dF^{n*}(x)$ to find $dF^{n*}(x) = e^{n\psi_n(s_n)} e^{-s_n x} dV_s^{n*}(x)$, and in particular

$$1 - F^n(x_n) = 1 - F^{n*}(x_n \sigma_n \sqrt{n}) = e^{n\psi_n(s)} \int_{x_n \sigma_n \sqrt{n}}^{\infty} e^{-sy} dV_s^{n*}(y) \, .$$

6.2. Second step

Choose s_n to depend on n (and thus indirectly on σ_n, x_n) so that $x_n \sigma_n \sqrt{n} = n \psi_n{}'(s_n)$, or equivalently

$$\psi_n{}'(s_n) = x_n \sigma_n / \sqrt{n} \, .$$

Since $\lim_{n\to\infty} \sup_{0 \le \zeta \le \overline{s}} \left| \psi_n{}''(\zeta) - \psi^2(\zeta) \right| = 0, \psi^2(\zeta) > 0$, $\sigma_n \to \sigma$ and $x_n / \sqrt{n} \to 0$ a solution in $[0, \overline{s}]$ exists for large enough n.

Lemma 6: *If Assumptions 1 and 2 of the Large Deviations Theorem hold then* $s_n \to 0$ and $\psi_n{}''(s_n) \to \psi^2(0) = \sigma^2$. *Also* $n s_n^3 \to 0$ *if and only if Assumption 4, that is,* $n^{-1/6} x_n \to 0$.

Proof: Because $\psi'(0) = EZ_i^n = 0$, and $\psi'(s_n) = x_n \sigma_n / \sqrt{n}$, by the mean value theorem we may write $\psi_n{}''(\zeta) s_n = x_n \sigma_n / \sqrt{n}$ where $\zeta \in [0, \overline{s}]$. Then

$$s_n = \frac{x_n \sigma_n}{\psi_n{}''(\zeta) \sqrt{n}} .$$

From the basic assumption that $\sigma_n \to \sigma$ and $x_n / \sqrt{n} \to 0$ it follows immediately that $s_n \to 0$, and so $\psi_n''(s_n) \to \psi^2(0)$ by $\lim_{n\to\infty} \sup_{0\le\zeta\le\bar{s}} \left|\psi_n''(\zeta) - \psi^2(\zeta)\right| \to 0$.

Now write

$$ns_n^3 = \left(\frac{n^{-1/6} x_n \sigma_n}{\psi_n''(\zeta)} \right)^3,$$

this gives the final result.

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6.3. Third step

Define the quantity A_n by replacing ${V_{s_n}}^{n*}$ in the expression from step 1

$$e^{n\psi_n(s_n)}\int_{x_n\sigma_n\sqrt{n}}^{\infty} e^{-s_n y}d{V_{s_n}}^{n*}(y)$$

by a normal with mean $n\psi_n{}'(s_n)$ and variance $n\psi_n{}''(s_n)$

$$A_n \equiv e^{n\psi_n(s_n)}\int_{x_n\sigma_n\sqrt{n}}^{\infty} e^{-s_n y}\frac{1}{\sqrt{2\pi}\sqrt{n\psi_n{}''(s_n)}}e^{-(1/2)(y-n\psi_n{}'(s_n))^2/n\psi_n{}''(s_n)}dy$$

We rewrite A_n in a more convenient form. Use the substitution

$$y = n\psi_n{}'(s_n) + t\sqrt{n\psi_n{}''(s_n)}$$

and the fact that the lower limit of integration $x_n\sigma_n\sqrt{n} = n\psi_n{}'(s_n)$ to find

$$A_n = e^{n[\psi_n(s_n)-\psi_n{}'(s_n)s_n]}\frac{1}{\sqrt{2\pi}}\int_0^{\infty} e^{-ts_n\sqrt{n\psi_n{}''(s_n)}-(1/2)t^2}dt$$

Complete the square in the numerator to get

$$A_n = e^{n[\psi_n(s_n)-\psi_n{}'(s_n)s_n+(1/2)\psi_n{}''(s_n){s_n}^2]}\left(1-\Phi(s_n\sqrt{n\psi_n{}''(s_n)})\right)$$

6.4. Fourth Step

Use Lemmas 4 and 5 to apply the Berry-Esseen Theorem to ${V_{s_n}}^{n*}$ and find for all y

$$\left|{V_{s_n}}^{n*}(y)-\Phi\left(\frac{y-n\psi_n{}'(s_n)}{\sqrt{n\psi_n{}''(s_n)}}\right)\right| < \frac{9M_{n3}}{\sqrt{n}\left[\psi_n{}''(s_n)\right]^{3/2}}$$

where M_{n3} is the third absolute central moment of V_{ns_n}.[5]

6.5. Fifth Step

How close is A_n to our target $1-F^n(x_n)$?

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[5] The parallel claim in Feller's proof is the related but different inequality $\left|V^{n*}(y)-\Phi\left(\frac{y-n\psi_n{}'(s)}{\sqrt{n\psi_n{}''(s)}}\right)\right| < \frac{9M_{n3}}{\sqrt{n}\sigma_n^3}$. This claim seems to be an incorrect application of the Berry-Esseen theorem which requires the variance of V_n rather than σ_n^2 in the denominator

$$
\begin{aligned}
&| 1 - F^n(x_n) - A_n | = \\
&\left| e^{n\psi_n(s_n)} \int_{x_n\sigma_n\sqrt{n}}^{\infty} e^{-s_n y} dV_{s_n}{}^{n*}(y) - e^{n\psi_n(s_n)} \int_{x_n\sigma\sqrt{n}}^{\infty} e^{-s_n y} \frac{1}{\sqrt{2\pi}\sqrt{n\psi_n''(s_n)}} e^{-(1/2)(y - n\psi_n'(s_n))^2 / n\psi_n''(s_n)} dy \right| \\
&\le \left| e^{n\psi_n(s_n)} \int_{x_n\sigma_n\sqrt{n}}^{\infty} e^{-s_n y} \left[dV_{s_n}{}^{n*}(y) - \frac{1}{\sqrt{2\pi}\sqrt{n\psi_n''(s_n)}} e^{-(1/2)(y - n\psi_n'(s_n))^2 / n\psi_n''(s_n)} dy \right] \right| \\
&= \left| e^{n\psi_n(s_n)} \int_{x_n\sigma_n\sqrt{n}}^{\infty} e^{-s_n y} \left[dV_{s_n}{}^{n*}(y) - \phi\left(\frac{y - n\psi_n'(s_n)}{\sqrt{n\psi_n''(s_n)}} \right) dy \right] \right|
\end{aligned}
$$

Integrate by parts to find

$$
\begin{aligned}
&| 1 - F^n(x_n) - A_n | \le \\
&e^{n\psi_n(s_n)} e^{-s_n x_n \sigma_n \sqrt{n}} \left| \Phi\left(\frac{x_n\sigma_n\sqrt{n} - n\psi_n'(s_n)}{\sqrt{n\psi_n''(s_n)}} \right) - V^{n*}(x_n\sigma_n\sqrt{n}) \right| \\
&+ e^{n\psi_n(s_n)} \int_{x_n\sigma_n\sqrt{n}}^{\infty} s_n e^{-s_n y} \left| V_{s_n}{}^{n*}(y) - \Phi\left(\frac{y - n\psi_n'(s_n)}{\sqrt{n\psi_n''(s_n)}} \right) \right| dy
\end{aligned}
$$

Now plug the bound from Step 4.

$$
\begin{aligned}
&| 1 - F^n(x_n) - A_n | \\
&\le e^{n\psi_n(s_n)} \frac{9M_{n3}}{\sqrt{n}\,[\psi_n''(s_n)]^{3/2}} \left[e^{-s_n x_n \sigma_n \sqrt{n}} + \int_{x_n\sigma_n\sqrt{n}}^{\infty} s_n e^{-s_n y} dy \right] \\
&= \frac{18M_{n3}}{\sqrt{n}\,[\psi_n''(s_n)]^{3/2}} e^{n\psi_n(s_n) - s_n x_n \sigma_n \sqrt{n}} \qquad (*) \\
&= \frac{18M_{n3}}{\sqrt{n}\,[\psi_n''(s_n)]^{3/2}} e^{n\psi_n(s_n) - n\psi_n'(s_n) s_n}
\end{aligned}
$$

where the last step follows from $x_n\sigma_n\sqrt{n} = n\psi_n'(s_n)$. Now from Step 2

$$
A_n = e^{n[\psi_n(s_n) - \psi_n'(s_n)s_n + (1/2)\psi_n''(s_n){s_n}^2]} \left(1 - \Phi(s_n\sqrt{n\psi_n''(s_n)})\right).
$$

By Lemma 3 we may replace the cumulative normal tale with the density and get the inequality

$$
A_n \ge e^{n[\psi_n(s_n) - \psi_n'(s)s_n + (1/2)\psi_n''(s_n){s_n}^2]} \frac{K}{\sqrt{2\pi}} (s_n\sqrt{n\psi_n''(s_n)})^{-1} e^{-\frac{1}{2}(s_n\sqrt{n\psi_n''(s_n)})^2}.
$$

Hence

$$
e^{n[\psi_n(s_n) - \psi_n'(s_n)s_n]} \le \frac{\sqrt{2\pi}}{K} s_n \sqrt{n\psi_n''(s_n)} A_n
$$

plug this into (*), and we now have

$$| 1 - F^n(x_n) - A_n \ | \\ \leq \frac{18 M_{n3}}{\sqrt{n}\,[\,\psi_n\,''(s_n)\,]^{3/2}} \frac{\sqrt{2\pi}}{K} s_n \sqrt{n\psi_n\,''(s_n)} A_{ns}$$

or dividing

$$| \frac{1 - F^n(x_n)}{A_{ns}} - 1 \,| \leq \frac{18 M_{n3}}{\psi_n\,''(s_n)} \frac{\sqrt{2\pi}}{K} s_n \,.$$

By Lemma 1 $s_n \to 0$ so the RHS goes to zero. Note that for this result we do not need $n^{-1/6} x_n \to 0$, $n^{-1/2} x_n \to 0$ would be sufficient.

6.6. Sixth and Final Step

We must now show

$$\frac{A_n}{1 - \Phi(x_n)} = e^{n[\psi_n(s_n) - \psi_n\,'(s_n)s + (1/2)\psi_n\,''(s_n){s_n}^2]} \frac{\left(1 - \Phi(s_n \sqrt{n\psi_n\,''(s_n)})\right)}{1 - \Phi(x_n)} \to 1$$

Will do this by showing that both

$$e^{n[\psi_n(s_n) - \psi_n\,'(s_n)s_n + (1/2)\psi_n\,''(s_n){s_n}^2]} \to 1$$

and

$$\frac{\left(1 - \Phi(s_n \sqrt{n\psi_n\,''(s_n)})\right)}{1 - \Phi(x_n)} \to 1\,.$$

6.6.1. Final Step First Half

$$e^{n[\psi_n(s_n) - \psi_n\,'(s_n)s_n + (1/2)\psi_n\,''(s_n){s_n}^2]} \to 1$$

or equivalently that

$$g_n(s_n) = n[\psi_n(s_n) - \psi_n\,'(s_n)s_n + (1/2)\psi_n\,''(s_n){s_n}^2] \to 0\,.$$

Observe that $g_n(0) = 0, g_n\,'(0) = 0, g_n\,''(0) = 0$. Hence by the mean value theorem

$$g_n(s) = (1/6) g_n\,'''(\zeta) n {s_n}^3\,.$$

By the uniform boundedness assumptions on the third through fifth derivatives of ψ_n $g_n\,'''(\zeta)$ is uniformly bounded, so by Lemma 1 $g_n(s_n) \to 0$ provided $n^{-1/6} x_n \to 0$.

6.6.2. Final Step Last Half

$$\frac{\left(1-\Phi(s_n\sqrt{n\psi_n''(s_n)})\right)}{1-\Phi(x_n)} \to 1$$

Use Lemma 2 and $x_n, s_n\sqrt{n\psi_n''(s_n)} \to \infty$ to conclude that

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\left(1-\Phi(s_n\sqrt{n\psi_n''(s_n)})\right)}{1-\Phi(x_n)} \\
&= \lim_{n\to\infty}\frac{\dfrac{(s_n\sqrt{n\psi_n''(s_n)})^{-1}\phi(s_n\sqrt{n\psi_n''(s_n)})}{1-\Phi\left(s_n\sqrt{n\psi_n''(s_n)}\right)}\left(1-\Phi(s\sqrt{n\psi_n''(s)})\right)}{\dfrac{{x_n}^{-1}\phi(x_n)}{1-\Phi(x_n)}(1-\Phi(x_n))} \\
&= \lim_{n\to\infty}\frac{x_n\phi(s_n\sqrt{n\psi_n''(s_n)})}{s_n\sqrt{n\psi_n''(s_n)}\phi(x_n)} \\
&= e^{-(1/2)[{s_n}^2 n\psi_n''(s_n)-{x_n}^2]}\frac{x_n}{s_n\sqrt{n\psi_n''(s_n)}} \\
&= e^{-(1/2)[s_n^2 n\psi_n''(s)-n[\psi_n'(s_n)]^2/{\sigma_n}^2]}\frac{\psi_n'(s_n)}{\sigma_n s_n\sqrt{\psi_n''(s_n)}}
\end{aligned}$$

Consider first by Lemma 1

$$\frac{\psi_n'(s_n)}{\sigma_n s_n\sqrt{\psi_n''(s_n)}} = \frac{\psi_n''(\zeta)s_n}{\sigma_n s\sqrt{\psi_n''(s_n)}} \to \frac{\psi^2(0)}{\sigma\sqrt{\psi^2(0)}} = 1$$

So we are left with showing

$$h_n(s_n) = n\left[s_n^2\psi_n''(s_n) - [\psi_n'(s_n)]^2/{\sigma_n}^2\right] \to 0\,.$$

Here $h_n(0) = 0, h_n'(0) = 0, h_n''(0) = 0$ so

$$h_n(s_n) = (1/6)h_n'''(\zeta)ns_n^3.$$

Again by the uniform boundedness assumptions on the third through fifth derivatives of ψ_n, $h_n'''(\zeta)$ is uniformly bounded, so by Lemma 1 this is true provided $n^{-1/6}x_n \to 0$.

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References

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