Decision Theory: Risk

Lotteries and Expected Utility

Luce, D. and H. Raiffa [1957]: *Games and Decisions,* John Wiley chapter 2.5

there are r prizes $1,\ldots,r$

a lottery L consists of a finite vector $(p_1,\ldots,p_r)$ where p_i is the "probability" of winning prize i

properties of "probabilities" $p_i \geq 0, \sum_{i=1}^{r} p_i = 1$

Definition: the lottery L_i has $p_i = 1$

Preferences $\geq$ are defined over the set of lotteries

order the lotteries so that $L_i \geq L_{i+1}$, that is higher numbered prizes are worse

Usual preference assumptions:

1) transitivity

2) continuity: for each L_i there exists a lottery $\tilde{L}_i$ such that $p_j = 0$ for $j = 2, \ldots, r-1$ and $L_i \sim \tilde{L}_i$

(in words: we can find probabilities of the best and worst prize that are indifferent to any lottery)

Definition: u_i is such that $\tilde{L}_i = (u_i, 0, \ldots, 0, (1-u_i))$

Assumptions relating to probability:

a compound lottery is a lottery in which the prizes are lotteries

we can write a compound lottery $(q^1, L^1, q^2, L^2, \ldots, q^k, L^k)$ where

q^i is the probability of lottery L^i (not to be confused with L_i)

1) reduction of compound lotteries

preferences are extended from simple lotteries to lotteries over lotteries by the usual laws of probability

example: $L^1 = (p_1^1, p_2^1, \ldots, p_r^1)$, $L^2 = (p_1^2, p_2^2, \ldots, p_r^2)$

$$(q_1, L^1, q_2, L^2) \sim (q^1 p_1^1 + q^2 p_1^2, q^1 p_2^1 + q^2 p_2^2, \ldots, q^1 p_r^1 + q^2 p_r^2)$$

2) substitutability (independence of irrelevant alternatives)

for any lottery L the compound lottery that replaces L_i with $\tilde{L}_i$ is indifferent to L

$$(p_1, p_2, \ldots, p_r) \sim (p_1, L_1, p_2, L_2, \ldots, p_i, \tilde{L}_i, \ldots, p_r, L_r)$$

3) monotonicity

$(p, 0, \ldots, 0, (1-p)) \geq (p', 0, \ldots, 0, (1-p'))$ if and only if $p \geq p'$

Expected utility theory:

Start with a lottery $L = (p_1, \ldots, p_r)$

Using transitivity and continuity L is indifferent to the compound lottery $(p_1\tilde{L}_1, \ldots, p_r\tilde{L}_r)$

Notice that the lotteries $\tilde{L}_i$ involve only the highest and lowest prizes

Now apply reduction of compound lotteries: this is equivalent to the lottery

$L \sim (u, 0, \ldots, 0, (1-u))$ where $u = \sum_{i=1}^{r} p_i u_i$

This says that we may compare lotteries by comparing their "expected utility" and by monotonicity, higher utility is better

Allais Paradox

Take Q = 1 billion dollars US

Decision problem 1:

Q for sure

(or)

.1 x 5Q, .89 x 1Q, .01 x 0Q

Decision problem 2:

.1 x 5Q, .9 x 0Q

(or)

.11 x 1Q, .89 x 0Q

Decision problem 1:

1 x 1Q for sure **[most common choice]**

(or)

.1 x 5Q, .89 x 1Q, .01 x 0Q

Decision problem 2:

.1 x 5Q, .9 x 0Q **[most common choice]**

(or)

.11 x 1Q, .89 x 0Q

So $u(1) > .1u(5) + .89u(1) + .01u(0)$ or $u(5) < 1.1u(1) - .1u(0)$

And $.1u(5) + .9u(0) > .11u(1) + .89u(0)$ or $u(5) > 1.1u(1) - .1u(0)$

Notice that the original problem had Q equal to 1 million US. This doesn’t work well anymore because most people make the second choice in the first problem and the first choice in the second problem, which is consistent with expected utility

Two views:

1) this is a big problem [Tversky and Kahneman, 1979]

decent theory due to Machina [1982], Segal [1990]

2) this is a curiousity due to the unusual magnitudes of the payoffs Rubsinsten [1988], Leland [1994]

Subjective Uncertainty

Ellsburg Paradox

Ellsberg [1961]

Two urns: each contains red balls and black balls

Urn 1: 100 balls, how many red or black is unknown

Urn 2: 50 red and 50 black

Choice 1: bet on urn 1 red or urn 2 red

Choice 2: bet on urn 1 black or urn 2 black

Urn 1: 100 balls, how many red or black is unknown

Urn 2: 50 red and 50 black

Choice 1: bet on urn 1 red or urn 2 red **[urn 2]**

Choice 2: bet on urn 1 black or urn 2 black **[urn 2]**

1 says that urn 2 red more likely than urn 1 red

2 says that urn 2 black more likely than urn 2 black

but this is inconsistent with probabilities that add up to 1

Can introduce theory of “ambiguity aversion” as in Schmeidler [1989],

Ghirardato and Marinacci [2000]

Basically probabilities do not add up to one; remaining probability is assigned to “nature” moving after you make a choice and choosing the worst possibility for you. [The stock market always tumbles right after I buy stocks.]

Ellsburg Paradox Paradox

we should be able to break the indifference

Urn 1: 1000 balls, how many red or black is unknown

Urn 2: 501 red and 499 black

Choice 1: bet on urn 1 red or urn 2 black **[urn 2]**

Choice 2: bet on urn 1 black or urn 2 black **[urn 2]**

Combine this into a single choice:

Bet on urn 1 red, urn 1 black or urn 2 black

Ambiguity aversion says go with urn 2 black…

But this is a bad idea: flip a coin to decide between urn 1 red and urn 1 black

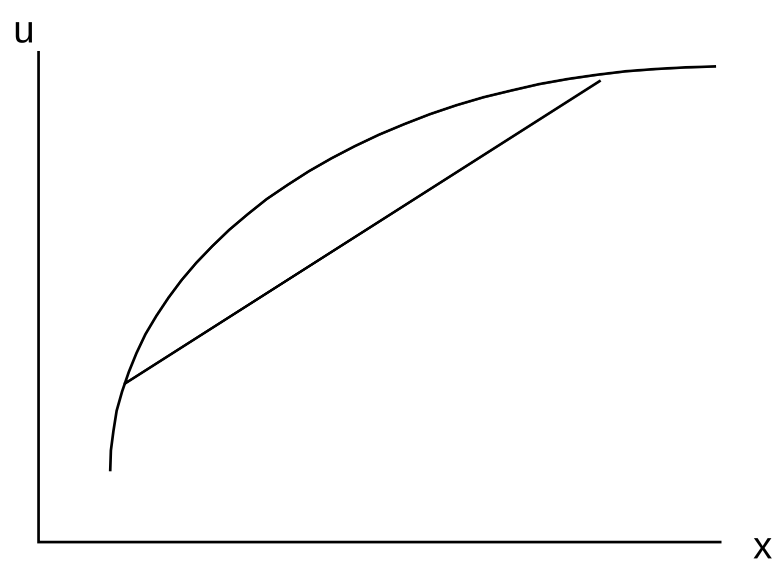
Risk Aversion

Jensen's inequality

u is a concave function if and only if $u(Ex) \geq Eu(x)$

that is: you prefer the certainty equivalent

so concavity = risk aversion



Risk premium

y a random income with $Ey = 0, Ey^2 = 1$

$u(x - p) = Eu(x + \sigma y)$

Taylor series expansion:

$$
\begin{aligned}
u(x) - pu'(x) &= E\left[u(x) + \sigma u'(x)y + (1/2)\sigma^2 u''(x)y^2\right] \\
&= u(x) + (1/2)\sigma^2 u''(x)
\end{aligned}
$$

so $p = -\dfrac{u''(x)}{u'(x)}\dfrac{\sigma^2}{2}$

we can also consider the relative risk premium

$u(x - \rho x) = Eu(x + \sigma y x)$

$$\rho = -\frac{u''(x)x}{u'(x)}\frac{\sigma^2}{2}$$

Measures of Risk Aversion

Absolute risk aversion

The coefficient of absolute risk aversion is $-\dfrac{u''(x)}{u'(x)}$

Relative risk aversion

The coefficient of relative risk aversion is $-\dfrac{u''(x)x}{u'(x)}$

Changes in Risk Aversion with Wealth

We ordinarily think of absolute risk aversion as declining with wealth (this is a condition on the third derivative of u).

Constant relative risk aversion

$u(x) = \frac{x^{1-\rho}}{1-\rho}$ also known as "constant elasticity of substitution" or CES

$\rho \geq 0$

$$-\frac{u''(x)x}{u'(x)} = \frac{\rho x^{-\rho-1}x}{x^{-\rho}} = \rho$$

$\rho = 0$ linear, risk neutral

$\rho = 1$ $u(x) = \log(x)$

useful for empirical work and growth theory

note that constant relative risk aversion implies declining absolute risk aversion

How risk averse are people?

Equity premium

Mehra and Prescott [1985]; Shiller [1989] data annual 1871-1984

Mean real return on bonds r_b=1.9%; Mean real return on S&P 7.5%

Equity premium $\lambda = .056$

Standard error of real stock return 18.1%, $\sigma = 0.181$.

normalized real per capita consumption standard error $s = .035$

let x denote initial wealth

Let α be fraction of portfolio in S&P

calculate consumption

$$u((1-\alpha)x(1+r_b)+\alpha x(1+\overline{r}_s+\sigma y))=$$

$$u(x+xr_b+\alpha\lambda x+\alpha\sigma yx)$$

$$\frac{d}{d\alpha}Eu(x+xr_b+\alpha\lambda x+\alpha\sigma yx)$$

$$=\lambda xEu'+\sigma xEyu'$$

$$=u'\lambda x+\lambda xEu''()[\alpha\sigma yx]+\sigma xEyu'(x+xr_b+\alpha\lambda x+0)+\sigma xEyu''()[\alpha\sigma yx]$$

$$=u'\lambda x+\alpha u''\sigma^2x^2=0$$

$$\rho=\lambda/(\alpha\sigma^2)\approx 1.81\alpha^{-1}$$

$$s^2=\mathrm{var}[((1-\alpha)x+\alpha(1+\lambda)x+\alpha\sigma yx)/x]=\alpha^2\sigma^2$$

or $\alpha^{-1}\approx\sigma/s=5.17$ giving $\rho=8.84$

Risk Aversion in the Laboratory

In laboratory experiments we often observe what appears to be risk averse behavior over small amount of money (typical payment rates are less than $50/hour, and play rarely lasts two hours)

How can people be risk averse over gambles involving such an insignificant fraction of wealth?

Rabin [2000]: Risk aversion in the small leads to impossible results in the large

"Suppose we knew a risk-averse person turns down 50-50 lose $100/gain $105 bets for any lifetime wealth level less than $350,000, but knew nothing about the degree of her risk aversion for wealth levels above $350,000. Then we know that from an initial wealth level of $340,000 the person will turn down a 50-50 bet of losing $4,000 and gaining $635,670."

Risk Aversion in the Field

There is surprisingly little systematic evidence about how risk averse people are.

One exception: Hans Binswanger [1978] took his grant money to rural India and conducted a series of experiments involving gambles for a significant fraction of annual income.

His findings: risk aversion is high (ρ on the order of 20), and inconsistent with expected utility theory – initial wealth plays a greater role than the theory allows, along much the same lines discussed by Rabin.

Remark: it is easy to see that deviations from the amount that is “expected to be earned” play some role. But it is a long leap from that to a systematic theory.