2016 - Problem set 2 - Community Enforcement

Social Norms and Community Enforcement. Kandori, Review of Economic Studies, 1992

Introduction We study how communities can enforce cooperative behaviors that are not incentive compatible in a static context (not static Nash). You can think of enforcing the cooperative outcome of the Prisonner Dilemma when N players of a community repeatedly play the $2-$player game matched in pairs every period. But the pairs ae formed each period according to a matching rule, and there is incomplete information with respect to the past behavior of current partners. In this context, how to enforce cooperation?

We wish to prove here Theorem 2 of Kandori 1992 that are the basis for community enforcement when the community is endowed with what is called "local information processing". We will see deterministic local information processing.

The Set-up

Set of Players $N = \{1, 2, \ldots, 2n\}$. It is partitioned in $N_1 = \{1, \ldots, n\}$ and $N_2 = \{n+1, \ldots, 2n\}$. Each player plays a stage game each period and each player's total payoff is the expected sum of his stage payoffs discounted by $\delta \in (0, 1)$.

Matching Process : $\mu(i, t) \equiv$ player's i match at time t

In each stage, each type-1 player is matched with a type-2 player according to the matching rule $\mu(.,.)$ and they play a 2-player stage game. We impose no structure on the matching rule (can be endogenous, history-dependent...)

Stage Game Payoffs Let A_i the finite action set of type i. Payoff function of the stage game:

$$g : A \to \mathbb{R}^2, \text{ with } A \equiv A_1 \times A_2$$

Minimax Payoffs We define the Minimax Payoffs in the following way: The Minimax point $M^1 \in A \equiv A_1 \times A_2$ for type-1 players is defined:

$$M_2^1 \in argmin_{a_2 \in A_2} \, [\max_{a_1 \in A_1} \; g_1(a_1, a_2)]$$

$$M_1^1 \in argmax_{a_1 \in A_1} \, [g_1(a_1, M_2^1)]$$

It means: player 2 wants to minimize the payoff of player 1, but knows that, whatever he does, player 1 will maximize, given the action chosen by player 2. Taking this into account, player 2 chooses the action that will minimize player 1's payoff, regardless of the impact on his own payoffs (it can hurt him too). Consequently, player 1 maximizes, given player 2's action.

- Mutual minimaxing point: $\left(M_1^2, M_2^1\right) \equiv m \equiv (m_1, m_2)$. So here, nobody is best-responding, they mutually minimize each other.
- Normalization: $g_1(M_1) = g_2(M_2) = 0$

Set of Payoffs

$$V \equiv \{v \in cog(A) | v >> 0\}$$

($cog(A)$ is the convex hull of $g(A)$).

Information Structure (Section 5 of the paper)

Definition 1 *A matching game with local information processing has the following information structure.*

1. *A state $z_i(t) \in Z_k$ is assigned to player $i \in N_k$ $(k = 1, 2)$ at t.*
2. *When player i and j meet at time t and take actions $(a_i(t), a_j(t))$, their next states are determined by*

$$\big(z_i(t+1), z_j(t+1)\big) = \tau\big(z_i(t), z_j(t), a_i(t), a_j(t)\big)$$

3. *At t, i can observe at least $(z_i(t), z_{\mu(i,t)}(t))$ before choosing his action.*

Equilibrium Concept: Sequential Equilibrium *(Kreps & Wilson 1982)*

Let $\mathcal{H}^{t-1}$ the set of all possible histories of play up to $t-1$. We are in a game of incomplete information, therefore players don't observe the full past history and they each observe different actions: they have private information. Let $\mathcal{H}^{i,t-1}$ the set of all possible histories of play up to $t-1$ in the information set of player i.

A *belief assesment* is a sequence $\mu = (\mu_{i,t})_{t\geq 1, i\in N}$ with $\mu_{i,t} : \mathcal{H}^{i,t} \to \Delta(\mathcal{H}^t)$, that is, given the private history h_i of player i, $\mu_{i,t}(h_i)$ is the probability distribution representing the belief that player i holds on the full history.

A *(pure) strategy* for player i is:

$$\sigma_i : \cup_{t\geq 0}\, \mathcal{H}^{i,t} \to \mathcal{A}_i$$

(Here I refer to the generic action set of a player i, $\mathcal{A}_i$. In our game, the action set is the same for all the players of a same type). That is, at each possible history of i's private information set, we have to define an action, be it a history on or off the equilibrium path.

A *Sequential Equilibrium* of the repeated game is a pair (σ, μ) where σ is a strategy profile ($\sigma = \times_{i\in I}\sigma_i$) and μ is a belief assesment such that: 1) for each player i and each history $h_i \in \cup_{t\geq 0}\, \mathcal{H}^{i,t}$, σ_i is a best reply in the continuation game, given the strategies of the other players and the belief that player i holds regarding the past; 2) the beliefs must be **consistent** "in the sense of Kreps-Wilson" (I don't define it here because we will not need it and I refer you to the KW (82) paper for more details).

However, Kandori wants to find equilibria that have some "nice" properties, among which:

Definition 2 *A sequential equilibrium in a matching game with local information is **straightforward** if, given that all other players'choice of actions depends only on their and their partners' labels, a player best response also depends only on his and his partner's labels, even if he had more information than those:*

$$a_i(t) = \sigma_i(z_i(t), z_{\mu(i,t)}(t))\ \forall i \in N$$

Theorem 2

Assumption 3 *$\exists r \in A$ such that:*

$$g_1(m_1, r_2) > g_1(m) \geq g_1(r_1, m_2)$$

$$g_2(r_1, m_2) > g_2(m) \geq g_2(m_1, r_2)$$

Theorem 4 (2) *Under the previous assumption, every point $v \in V$ is sustained by a **straightforward** and **globally stable** equilibrium with local information processing, if $\delta \in (\delta^*, 1)$ for some δ^*, which is **independent of the matching rule** and **the population size**. Furthermore, only 3 actions are prescribed to each player.*

Let $v \in V$, and let a^* the action profile to achieve the payoff v.

Candidate Equilibrium We study the following candidate equilibrium.

State Space (that is, the set of possible labels for each type of player)

$$Z_1 = Z_2 = Z = \{0, 1, \dots, T\}$$

where 0 means "innocent" and any other label means "guilty".

Individual Strategy (symmetric within types $k = 1, 2$). If two innocent players are matched, they choose the designated action a^*. If two guilty players meet, they mutually minimax each other. If an innocent player encounters a guilty player, the former minimaxes the latter but the latter chooses the "repenting" action r defined in (Al).

$\forall z \in Z \times Z$

$$\sigma(z) = \begin{cases} a^* & \text{if } z = (0,0) \\ (m_1, r_2) & \text{if } z_1 = 0, z_2 \neq 0 \\ (r_1, m_2) & \text{if } z_1 \neq 0, z_2 = 0 \\ m & \text{if } z_1, z_2 \neq 0. \end{cases}$$

State Transition The state transition obeys a simple rule; any deviation starts a T-period punishment. For type 1 players:

$\forall z \in Z \times Z, \forall a \in A$

$$\tau_1(z, a) = \begin{cases} 0 & \text{if } z_1 = 0 \text{ and } a_1 = \sigma_1(z) \\ z_1 + 1 \pmod{T+1} & \text{if } z_1 \neq 0 \text{ and } a_1 = \sigma_1(z) \\ 1 & \text{if } a_1 \neq \sigma_1(z), \end{cases}$$

(symmetric for type-2 players).

Remember the definition of *sequential equilibrium*, we have to check that strategies are optimal $\forall h^{i,t} \in \mathcal{H}^{i,t}, \forall t, \forall i$.

Incentives when type-1 is guilty We start in period t, $t = 1, 2, \dots.$ from any possible history, where player i is guilty (that is $z_i(t) > 0$).

Question 1 Assume player i of type 1 has a guilty label. Given that we start from any $h^{t-1} \in \mathcal{H}^{t-1}$ such that $z_i(t) > 0$, we cannot impose anything on the other players labels. Assuming that all the other players follow the equilibrium strategies, *can you tell what will be the share of guilty type 2 players at $t + T$?*

Give a lower bound $\underline{V}^g$ on player i's continuation payoff as a function of v_1 (the payoff that the candidate equilibrium aims at sustaining) and $x(t)$ defined in the following way:

$\forall t = 1, 2, \dots, \forall i = 1, 2, \dots, n$

$$x(t) = \begin{cases} g_1(m) & \text{if} \quad z_{\mu(i,t)}(t) \neq 0 \\ g_1(r_1, m_2) & \text{if} \quad z_{\mu(i,t)}(t) = 0 \end{cases}$$

when he sticks to the equilibrium strategy.

Question 2 Give a upper bound $\bar{V}^g_D$ on player i's continuation payoff as a function of v_1, $g(.), m, r$, if player i deviates once from equilibrium play when guilty. (Remember the Principle of Dynamic Programming (Abreu 1988) that says that we only have to check one-shot deviations).

Question 3 Write the Incentive Compatibility Constraint for the Guilty player

$$\underline{V}^g \geq \bar{V}_D^g \tag{IC^g}$$

(that is, the worst he can do by sticking to equilibrium play is better than the best he can do by deviating) and prove that a sufficient condition for (IC^g) to hold is:

$$(1-\delta^T)g_1(r_1, m_2) + \delta^T v_1 \geq 0 \tag{$*$}$$

Incentives when type-1 is innocent We start in period t, $t = 1, 2, \ldots.$ from any possible history where player i is innocent (that is $z_i(t) = 0$).

Question 4

Give a lower bound $\underline{V}^I$ on player i's continuation payoff as a function of v_1, $g(.), m, r$ and v_1^* with

$$v_1^* = \max_{a \in A} g_1(a)$$

when he sticks to the equilibrium strategy.

Question 5 Give a upper bound $\bar{V}_D^I$ on player i's continuation payoff as a function of v_1, $g(.), m, r$, if player 1 deviates once from equilibrium play when Innocent.

Question 6 Write the Incentive Compatibility Constraint for the Innocent player

$$\underline{V}^I \geq \bar{V}_D^I \tag{IC^I}$$

Finding δ^* and T Note that if we find a δ^* and a T independent from the matching rule and the population size, we have proved that our candidate equilibrium is a sequential equilibrium of the local information processing game.

Question 7 Satisfying $(*)$. What is the sign of $g_1(r_1, m_2)$? And of v_1? Considering the LHS of $(*)$ as a function of δ^T, can you say if this function is increasing? Decreasing? What happens when $\delta^T \to 1$? Can $(*)$ be satisfied for some δ^T?

Question 8 Prove that keeping δ^T constant but increasing δ, we can find a δ such that (IC^I) is satisfied. Call it δ^*. Does it depend on the matching rule? On the population size?

Question 9 (complementary) Is the equilibrium *straightforward*? is it *globally stable*?
Remember the definition:

Definition 5 *An equilibrium sustaining payoffs $v \in V$ is globally stable if for any given finite history of actions h,*

$$\lim_{t \to \infty} E(v_i(t)|h) = v_k, \quad \forall i \in N_k, k = 1, 2$$

where $v_i(t)$ is player i's continuation payoffs at t and $E(.|h)$ is the conditional expectation.