Competitive Equilibrium with Pure Exchange

Economic Fundamentals

n traders

k goods

x_j^i consumption by trader i of good j

x^i denote the vector, bundle or basket of goods consumed by trader j

trader i's preferences for consuming different goods given by her utility function $u^i(x^i)$

trader i endowed with $\bar{x}_j^i$ of good j

- there is no production in this economy, it is a *pure exchange* economy
- traders simply exchange goods with each other
- the economy lasts only one period

Market Institutions

we assume "the law of one price"

traders scope out opportunities to such an extent that each good is sold (and purchased) at only one price

p_j the price of good j

p list of all prices of all goods, or the price vector

we assume **competitive behavior** traders do not perceive that they have any influence over market prices

theory of the result of trading in this economy: *competitive equilibrium*

(competitive equilibrium is not the Nash equilibrium of the competitive game…)

competitive equilibrium prices written as $\hat{p}$ are (by definition) prices at which every trader can simultaneously satisfy her desire to trade at those prices

Demand

$x_j^i(p,m)$ demand by trader i for good j when prices are p and money income is m

the solution to the problem

$\max_{x^i} u^i(x^i)$

subject to $\sum_{j=1}^{k} p_j x_j^i \leq m$ or $p \cdot x^i \leq m$

Excess Demand

in pure exchange economy money income generated by selling endowment

[doesn't cost anything extra to sell your endowment then buy it back, since the prices at which you buy and sell are the same]

demand to buy or net, or excess demand

$$z_j^i(p,\overline{x}^i) \equiv x_j^i(p,\sum_{j=1}^{m} p_j \cdot \overline{x}_j^i) - \overline{x}_j^i.$$

this can be negative, as big as $-\overline{x}_j^i$

Cobb Douglas Example

$$u(x_1, x_2) = Ax_1^{\alpha} x_2^{\beta}$$

demand from Lagrangean

$$Ax_1^{\alpha} x_2^{\beta} - \lambda(p_1 x_1 + p_2 x_2)$$

first order conditions

$$A\alpha x_1^{\alpha-1} x_2^{\beta} - \lambda p_1 = 0$$

$$A\beta x_1^{\alpha} x_2^{\beta-1} - \lambda p_2 = 0$$

rearrange and divide

$$\frac{A\alpha x_1^{\alpha-1} x_2^{\beta}}{A\beta x_1^{\alpha} x_2^{\beta-1}} = \frac{\lambda p_1}{\lambda p_2} \text{ cancelling terms } \frac{\alpha x_2}{\beta x_1} = \frac{p_1}{p_2}$$

$$\frac{\alpha x_2}{\beta x_1} = \frac{p_1}{p_2}$$ cross multiply $\alpha p_2 x_2 = \beta p_1 x_1$

plug into the budget constraint $p_1 x_1 + p_2 x_2 = m$ to get

$$p_1 x_1 + (\beta / \alpha) p_1 x_1 = m$$

or $x_1 = \frac{\alpha}{\alpha + \beta} \frac{m}{p_1}$, $x_2 = \frac{\alpha}{\alpha + \beta} \frac{m}{p_2}$

excess demand

$$z_1 = \frac{\alpha}{\alpha + \beta} \frac{p_1 \bar{x}_1 + p_2 \bar{x}_2}{p_1} - \bar{x}_1, \; x_2 = \frac{\beta}{\alpha + \beta} \frac{p_1 \bar{x}_1 + p_2 \bar{x}_2}{p_2} - \bar{x}_2$$

for simplicity take $\alpha + \beta = 1$ then

$$z_1 = \alpha \frac{p_2}{p_1} \bar{x}_2 - (1 - \alpha) \bar{x}_1, \; z_2 = (1 - \alpha) \frac{p_1 \bar{x}_1}{p_2} - \alpha \bar{x}_2$$

Aggregate or Market Excess Demand

$$z_j(p) \equiv \sum_{i=1}^{n} z_j^i(p)$$

in market for each good demand to buy cannot exceed zero

there is no production or outsider to provide supply to the market

one traders excess demand must be anothers excess supply

$\hat{p}$ competitive equilibrium prices are determined by

$z_j(\hat{p}) \leq 0$ for every good $j = 1, 2, \ldots, k$

Cobb-Douglas Economy

Two consumers both have identical preferences with $\alpha = 1/2$

Endowments are $\bar{x}^1 = (1,0), \bar{x}^2 = (0,1)$

$$z_1^1 = -\frac{1}{2}$$

$$z_1^2 = \frac{1}{2}\frac{p_2}{p_1} \text{ so } z_1 = -\frac{1}{2} + \frac{1}{2}\frac{p_2}{p_1}$$

$$z_2^1 = \frac{1}{2}\frac{p_1}{p_2}$$

$$z_2^2 = -\frac{1}{2} \text{ so } z_2 = \frac{1}{2}\frac{p_1}{p_2} - \frac{1}{2}$$

we aren't going to solve this yet

Properties of Demand

Individual demand: two key properties

Homogeneous of degree zero

$$x_j^i(\lambda p, \lambda m) = x_j^i(p, m)$$

(relationship to inflation, dollars versus quarters)

$$x_1 = \frac{\alpha}{\alpha + \beta} \frac{m}{p_1}$$

Satisfies the budget constraint

$$\sum_{j=1}^{k} p_j x_j^i(p_j, m) = m$$

$$p_1 \frac{\alpha}{\alpha + \beta} \frac{m}{p_1} + p_2 \frac{\beta}{\alpha + \beta} \frac{m}{p_2} = ??$$

Individual excess demand: two key properties

Homogeneous of degree zero

$$z_j^i(\lambda p) = z_j^i(p)$$

$$z_1 = \alpha \frac{p_2}{p_1}\bar{x}_2 - (1-\alpha)\bar{x}_1$$

Walras's Law

$$\sum_{j=1}^{k} p_j z_j^i(p) = 0$$

proof:

$$\begin{aligned} m &= \sum_{j=1}^{k} p_j x_j^i(p_j, m) \\ &= \sum_{j=1}^{k} p_j x_j^i(p_j, p \cdot \bar{x}^i) - p_j \bar{x}_j^i + p_j \bar{x}_j^i \\ &= \sum_{j=1}^{k} p_j z_j^i(p_j) + \sum_{j=1}^{k} p_j \bar{x}_j^i \\ &= \sum_{j=1}^{k} p_j z_j^i(p_j) + m \end{aligned}$$

Aggregate excess demand: two key properties

Homogeneous of degree zero

$$z_j(\lambda p) = z_j(p)$$

$$z_1 = -\frac{1}{2} + \frac{1}{2}\frac{p_2}{p_1}$$

Walras's Law

$$\sum_{j=1}^{k} p_j z_j(p) = 0$$

$$p_1\left[-\frac{1}{2} + \frac{1}{2}\frac{p_2}{p_1}\right] + p_2\left[\frac{1}{2}\frac{p_1}{p_2} - \frac{1}{2}\right] = ??$$

Solving for Equilibrium

there are k different excess demand conditions $z_j(p) = 0$ and there are k different prices $p_1, \ldots, p_k$

but one excess demand condition is redundant

suppose $z_j(p) = 0$ for $j = 1, \ldots, k-1$, then from Walras's law $z_k(p) = 0$

on the other hand, if $z_j(p) = 0$ for all $j = 1, \ldots, k$ then so does $z_j(\lambda p) = 0$

so many competitive equilibria

can solve only for relative prices using $k-1$ excess demand equations

The Numeraire

may arbitrarily set the price of one good to 1

called the numeraire good, all prices are measured relative to that good

(for example – money is numeraire)

Cobb-Douglas Example

Pick one equation

$$z_1 = -\frac{1}{2} + \frac{1}{2}\frac{p_2}{p_1} = 0$$

so $p_2 / p_1 = 1$

pick the other equation

$$z_2 = \frac{1}{2}\frac{p_1}{p_2} - \frac{1}{2}$$

get the same answer of course

if we choose good 1 as numeraire then we have $p_1 = 1, p_2 = 1$

how do we find individual demands?

The First Welfare Theorem

Suppose we have a competitive equilibrium with prices p and individual demands x_i^j

is this pareto efficient?

That is: can we find $\tilde{x}_j^i$ socially feasible that makes nobody worse off and at least one person better off?

That is: can we find $\sum_{i=1}^n \tilde{x}_j^i \leq \sum_{i=1}^n \overline{x}_j^i$ so that $u^i(\tilde{x}^i) \geq u^i(\overline{x}^i)$ for everybody (all i) and for somebody (some i) $u^i(\tilde{x}^i) > u^i(\overline{x}^i)$?

Observation: if $u^i(\tilde{x}^i) > u^i(\overline{x}^i)$ then $p \cdot \tilde{x}^i > p \cdot \overline{x}^i$

Why??

Further observation: $u^i(\tilde{x}^i) \geq u^i(\overline{x}^i)$ then $p \cdot \tilde{x}^i \geq p \cdot \overline{x}^i$ (otherwise spend your extra income to buy more)

Our conclusion: if $u^i(\tilde{x}^i) \geq u^i(\bar{x}^i)$ for everybody (all i) and for somebody (some i) $u^i(\tilde{x}^i) > u^i(\bar{x}^i)$, then

$p \cdot \tilde{x}^i \geq p \cdot \bar{x}^i$ for all i and $p \cdot \tilde{x}^i > p \cdot \bar{x}^i$ for some i

add these together:

$$\sum_{i=1}^n p \cdot \tilde{x}^i > \sum_{i=1}^n p \cdot \bar{x}^i$$

on the other hand

$\sum_{i=1}^n \tilde{x}_j^i \leq \sum_{i=1}^n \bar{x}_j^i$, so adding over different goods

$$\sum_{j=1}^k p_j \sum_{i=1}^n \tilde{x}_j^i \leq \sum_{j=1}^k p_j \sum_{i=1}^n \bar{x}_j^i$$

which says that $\sum_{i=1}^n p \cdot \tilde{x}^i \leq \sum_{i=1}^n p \cdot \bar{x}^i$

- relationship to the core
- implications for international trade
- the edgeworth box
- the second welfare theorem
- the competitive mechanism

Finance

Trade in period 0 claims to consumption in period 1

k different states of nature in period 1, probability of state j is π_j

consumption in state j is c_j

time 0 price of consumption in state j is p_j

budget constraint $\sum_{j=1}^{k} p_j c_j \leq m$

"martingale" prices $\tilde{p}_j = p_j / \pi_j$

budget constraint in martingale prices

$E\tilde{p}c = \sum_{j=1}^{k} \pi_j \tilde{p}_j c_j \leq m$

Securities

Security a pays r_j in state j

Examples:

Arrow security on state j pays 1 in state j 0 in all other states

Price of an arrow security p_j or $\tilde{p}_j$

Bond pays 1 in all states

Price of a bond $\sum_{j=1}^{k} p_j$ or $E\tilde{p}$

"arbitrage pricing" = law of one price

Spanning

Two states $j = 1,2$, $k = 2$

Stock 1 pays $(2,1)$ price q_1, Stock 2 pays $(1,2)$ price q_2

What is the price of a bond?

Buy both stocks: get $(3,3)$ bond pays $(1,1)$ so 1/3rd of both stocks

$q_b = (q_1 + q_2) / 3$

what does spanning mean?

k different assets that are "independent"

can determine all asset prices in terms of a spanning set of assets

Short Sales

Stock 1 pays $(2,1)$ price q_1, Stock 2 pays $(3,1)$ price q_2

What is the price of a bond?

$$a(2,1) + b(3,1) = (1,1)$$

$$\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= (-1) \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

check $2(2,1) - 1(3,1) = (4,2) - (3,1) = (1,1)$

so buying 2 units of stock 1 and -1 units of stock 2 is the same as buying a bond: cost $2q_1 - q_2$

what does it mean to buy -1 unit of stock 2?

observation $2q_1 - q_2 > 0$ so we can conclude that $q_1 > q_2 / 2$

The Holdup Problem

“entrepreneur” (inventor, merchant) creates value of ρ

ρ is drawn from a uniform distribution over $[0,1]$ and is private information to the entrepreneur

case 1: the innovator receives a fraction of the social total $\phi\rho$

case 2: the innovator receives the entire social total ρ but must pay N existing “rights holders” for the right to create value

examples:

the silk road

patents and copyrights

pollution

efficiency = the good is always produced

in case 1 the good is always produced

in case 2

rights holder i set price p_i for his right and gets an expected revenue of

$$\left(1-(N-1)p-p_i\right)p_i$$

$p=1/(N+1)$

entrepreneur pays $\frac{N}{N+1}$ to clear the needed rights, so creation if
$\frac{N}{N+1}<\rho$

what happens as $N\to\infty$

as technologies grow more and more complex requiring more and more specialized inputs,monopoly power induced by patents and copyright becomes more and more socially damaging