1. We can solve the consumer's problem:

$$\max_{\{x_1, x_2\}} \quad \log x_1 + \log x_2$$
$$\text{s.t.} \quad p_1 x_1 + p_2 x_2 = M$$

$$L = \log x_1 + \log x_2 + \lambda (M - p_1 x_1 - p_2 x_2)$$

FOC:

$$\left.\begin{aligned} \frac{\partial L}{\partial x_1} = 0 &\Rightarrow \frac{1}{x_1} = \lambda p_1 \\ \frac{\partial L}{\partial x_2} = 0 &\Rightarrow \frac{1}{x_2} = \lambda p_2 \end{aligned}\right\} \Rightarrow p_1 x_1 = p_2 x_2$$

$$\Rightarrow p_1 x_1 = p_2 x_2 = \frac{M}{2} \Rightarrow \begin{cases} x_1(p_1, p_2, M) = \dfrac{M}{2p_1} \\ x_2(p_1, p_2, M) = \dfrac{M}{2p_2} \end{cases}$$

If the price of flounder, p_1, rises by 10%, we can define $\bar{p}_1 = 1.1 p_1$, as the new price.
then, the new demand of flounder would be:

$$x_1(\bar{p}_1, p_2, M) = \frac{M}{2\bar{p}_1} = \frac{M}{2.2 \times p_1}$$

So the ratio of demand would be:

$$\frac{x_1(\bar{p}_1, p_2, M)}{x_1(p_1, p_2, M)} = \frac{1}{1.1} = \frac{10}{11}$$

So the demand of flounder would decrease $\frac{1}{11}$.

$x_2(p_1, p_2, M) = \frac{M}{2p_2}$, so the demand for flour is not a function of p_1, which implies the change of p_1 has no effect on the demand for flour.

2. We need to check whether the functions satisfy:
   ① homogeneous of degree 0
   ② budget constraint.

ⓐ. $x_1 = \frac{m p_2}{p_1}$

$x_1(\lambda p_1, \lambda p_2, \lambda M) = \frac{\lambda M \cdot \lambda p_2}{\lambda p_1} = \lambda \cdot \frac{M p_2}{p_1} = \lambda x_1(p_1, p_2, M)$

So, this function does not satisfy ①.
it is not a demand function.

ⓑ $x_1 = x_2 = \frac{m}{p_1 + p_2}$

$x_1(\lambda p_1, \lambda p_2, \lambda M) = x_2(\lambda p_1, \lambda p_2, \lambda M) = \frac{\lambda M}{\lambda p_1 + \lambda p_2}$

$= \frac{M}{p_1 + p_2} = x_1(p_1, p_2, M) = x_2(p_1, p_2, M)$

So, it satisfies ①.

$p_1 x_1 + p_2 x_2 = \frac{p_1 M}{p_1 + p_2} + \frac{p_2 M}{p_1 + p_2} = M$

So, it satisfies ②.

$x_1 = x_2 = \frac{M}{p_1 + p_2}$ is demand function

Note: when the utility function is Leontief utility funct
$u(x_1, x_2) = \min\{x_1, x_2\}$
we would have the demand functions as
$x_1 = x_2 = \frac{M}{p_1 + p_2}$

③

③ Orca is consumer 1.

$$\max \ (x_1^1 x_2^1)^{\frac{1}{2}}$$
$$\text{s.t.} \quad p_1 x_1^1 + p_2 x_2^1 = p_2$$

$$L = (x_1^1 x_2^1)^{\frac{1}{2}} + \lambda (p_2 - p_1 x_1^1 - p_2 x_2^1)$$

FOC:
$$\left.\begin{aligned} \frac{\partial L}{\partial x_1^1} = 0 &\Rightarrow \frac{1}{2}\left(\frac{x_2^1}{x_1^1}\right)^{\frac{1}{2}} = \lambda p_1 \\ \frac{\partial L}{\partial x_2^1} = 0 &\Rightarrow \frac{1}{2}\left(\frac{x_1^1}{x_2^1}\right)^{\frac{1}{2}} = \lambda p_2 \end{aligned}\right\}$$

$$\Rightarrow \frac{x_2^1}{x_1^1} = \frac{p_1}{p_2} \Rightarrow \begin{cases} x_2^1 = \frac{1}{2} \\ x_1^1 = \frac{p_2}{2p_1} \end{cases}$$

Dorca is consumer 2.

$$\max \ (x_1^2 x_2^2)^{\frac{1}{2}}$$
$$\text{s.t.} \quad p_1 x_1^2 + p_2 x_2^2 = p_1$$

$$L = (x_1^2 x_2^2)^{\frac{1}{2}} + \mu (p_1 - p_1 x_1^2 - p_2 x_2^2)$$

FOC:
$$\left.\begin{aligned} \frac{\partial L}{\partial x_1^2} = 0 &\Rightarrow \frac{1}{2}\left(\frac{x_2^2}{x_1^2}\right)^{\frac{1}{2}} = \mu p_1 \\ \frac{\partial L}{\partial x_2^2} = 0 &\Rightarrow \frac{1}{2}\left(\frac{x_1^2}{x_2^2}\right)^{\frac{1}{2}} = \mu p_2 \end{aligned}\right\}$$

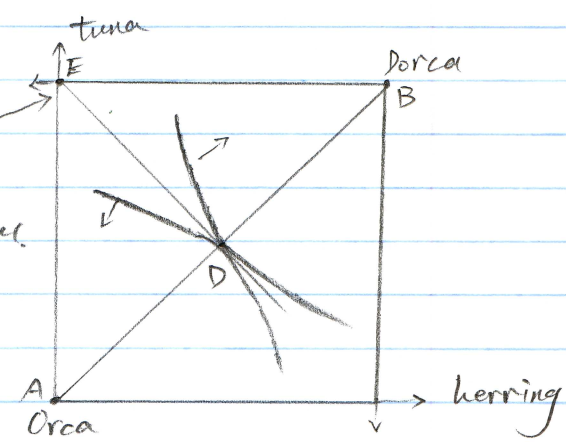
$$\Rightarrow \frac{x_2^2}{x_1^2} = \frac{p_1}{p_2} \Rightarrow \begin{cases} x_1^2 = \frac{1}{2} \\ x_2^2 = \frac{p_1}{2p_2} \end{cases}$$

Solve market clearing:

$$\begin{cases} x_1^1 + x_1^2 = 1 & \Rightarrow \frac{p_1}{p_2} = 1 \\ x_2^1 + x_2^2 = 1 & \Rightarrow \frac{p_1}{p_2} = 1. \end{cases}$$

So the demand is: $x_1^1 = x_2^1 = x_1^2 = x_2^2 = \frac{1}{2}$.

Edgeworth box:



point E is the endowment.
point D is the competitive equilibrium
line ED is the price line.

Line AB is the contract curve.
line AB is also the core.

#. Note: Any point on the contract curve is Pareto optimal.
Any point in the core is: ① pareto optimal; ② pareto dominates the endowment point.

4. We solve the case $\alpha = 1$ first; this case has some interesting properties.

Orca

$$\max \quad x_1^1 + x_2^1$$
$$\text{s.t.} \quad p_1 x_1^1 + p_2 x_2^1 = \bar{x}_1^1 \cdot p_1$$

$$\begin{cases} \text{if } p_1 > p_2 \Rightarrow x_2^1 = \dfrac{\bar{x}_1^1 \cdot p_1}{p_2}, \ x_1^1 = 0 \\ \text{if } p_1 < p_2 \Rightarrow x_2^1 = 0, \ x_1^1 = \bar{x}_1^1 \\ \text{if } p_1 = p_2 \Rightarrow x_1^1 + x_2^1 = \bar{x}_1^1 \end{cases}$$

Dorca

$$\max \quad x_1^2 + x_2^2$$
$$\text{s.t.} \quad p_1 x_1^2 + p_2 x_2^2 = p_2 \cdot \bar{x}_2^2$$

$$\text{if } p_1 > p_2, \Rightarrow x_1^2 = 0, \ x_2^2 = \bar{x}_2^2$$
$$\text{if } p_1 < p_2 \Rightarrow x_1^2 = \frac{p_2}{p_1} \bar{x}_2^2, \ x_2^2 = 0$$
$$\text{if } p_1 = p_2 \Rightarrow x_1^2 + x_2^2 = \bar{x}_2^2$$

So the excess demand would be:

Orca:

$$\text{if } p_1 > p_2, \begin{cases} z_1^1 = x_1^1 - \bar{x}_1^1 = -\bar{x}_1^1 \\ z_2^1 = x_2^1 - \bar{x}_2^1 = \dfrac{\bar{x}_1^1 \cdot p_1}{p_2} \end{cases}$$

$$\text{if } p_1 < p_2 \begin{cases} z_1^1 = x_1^1 - \bar{x}_1^1 = 0 \\ z_2^1 = x_2^1 - \bar{x}_2^1 = 0 \end{cases}$$

$$\text{if } p_1 = p_2 \begin{cases} z_1^1 = x_1^1 - \bar{x}_1^1 \\ z_2^1 = \bar{x}_1^1 - x_1^1 \end{cases}$$

Dorca:

$$\text{if } p_1 > p_2, \begin{cases} z_1^2 = x_1^2 - \bar{x}_1^2 = 0 \\ z_2^2 = x_2^2 - \bar{x}_2^2 = 0 \end{cases}$$

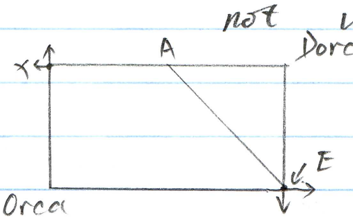
$$\text{if } p_1 < p_2 \begin{cases} z_1^2 = \dfrac{p_2}{p_1} \bar{x}_2^2 \\ z_2^2 = -\bar{x}_2^2 \end{cases}$$

$$\text{if } p_1 = p_2 \begin{cases} z_1^2 = x_2^2 - \bar{x}_2^2 \\ z_2^2 = \bar{x}_2^2 - x_2^2 \end{cases}$$

The competitive equilibrium in this case is: $p_1 = p_2$, and the demands satisfy:

$$\begin{cases} x_1^1 + x_1^2 = \bar{x}_1^1 \\ x_1^1 + x_2^1 = \bar{x}_1^1 \\ x_2^1 + x_2^2 = \bar{x}_2^2 \\ x_1^2 + x_2^2 = \bar{x}_2^2 \end{cases}$$

Note: in this case, the competitive equilibrium quantity is not unique. Increase $\bar{x}_2^2$ has no effect on p_2.



point E is endowment point,
line AE is competitive equilibrium and core
The whole set is "contract curve"

For $\alpha \neq 1$.

Orca

$$\max \quad (x_1^1)^\alpha + (x_2^1)^\alpha$$
$$\text{s.t.} \quad p_1 x_1^1 + p_2 x_2^1 = p_1 \bar{x}_1^1$$

$$L = (x_1^1)^\alpha + (x_2^1)^\alpha + \lambda(p_1 \bar{x}_1^1 - p_1 x_1^1 - p_2 x_2^1)$$

$$\left.\begin{array}{l} \frac{\partial L}{\partial x_1^1} = 0 \Rightarrow \alpha (x_1^1)^{\alpha-1} = \lambda p_1 \\ \frac{\partial L}{\partial x_2^1} = 0 \Rightarrow \alpha (x_2^1)^{\alpha-1} = \lambda p_2 \end{array}\right\}$$

$$\Rightarrow \begin{cases} x_1^1 = \dfrac{\bar{x}_1^1 p_1^{\frac{\alpha}{\alpha-1}} \cdot p_2^{\frac{1}{1-\alpha}}}{p_1^{\frac{\alpha}{\alpha-1}} \cdot p_2^{\frac{1}{1-\alpha}} + p_2} \\ x_2^1 = \dfrac{\bar{x}_1^1 p_1}{p_1^{\frac{\alpha}{\alpha-1}} p_2^{\frac{1}{1-\alpha}} + p_2} \end{cases}$$

$$\Rightarrow \begin{cases} z_1^1 = x_1^1 - \bar{x}_1^1 = -\dfrac{p_2 \cdot \bar{x}_1^1}{p_1^{\frac{\alpha}{\alpha-1}} p_2^{\frac{1}{1-\alpha}} + p_2} \\ z_2^1 = x_2^1 - \bar{x}_2^1 = \dfrac{\bar{x}_1^1 p_1}{p_1^{\frac{\alpha}{\alpha-1}} p_2^{\frac{1}{1-\alpha}} + p_2} \end{cases}$$

Dorca

$$\max \quad (x_1^2)^\alpha + (x_2^2)^\alpha$$
$$\text{s.t.} \quad p_1 x_1^2 + p_2 x_2^2 = p_2 \bar{x}_2^2$$

With the same way, we can solve:

$$\begin{cases} x_1^2 = \dfrac{p_1^{\frac{1}{\alpha-1}} p_2^{\frac{2-\alpha}{1-\alpha}} \cdot \bar{x}_2^2}{p_1^{\frac{\alpha}{\alpha-1}} p_2^{\frac{1}{1-\alpha}} + p_2} \\ x_2^2 = \dfrac{p_2 \bar{x}_2^2}{p_1^{\frac{\alpha}{\alpha-1}} p_2^{\frac{1}{1-\alpha}} + p_2} \end{cases}$$

$$\Rightarrow \begin{cases} z_1^2 = x_1^2 - \bar{x}_1^2 = \dfrac{p_1^{\frac{1}{\alpha-1}} p_2^{\frac{2-\alpha}{1-\alpha}} \cdot \bar{x}_2^2}{p_1^{\frac{\alpha}{\alpha-1}} p_2^{\frac{1}{1-\alpha}} + p_2} \\ z_2^2 = x_2^2 - \bar{x}_2^2 = \dfrac{p_1^{\frac{\alpha}{\alpha-1}} p_2^{\frac{1}{1-\alpha}} \cdot \bar{x}_2^2}{p_1^{\frac{\alpha}{\alpha-1}} p_2^{\frac{1}{1-\alpha}} + p_2} \end{cases}$$

For competitive equilibrium, $z_1^1 + z_1^2 = 0 \Rightarrow \frac{p_2}{p_1} = \left(\frac{\bar{x}_1^1}{\bar{x}_2^2}\right)^{1-\alpha}$

Good 1 is numeraire $\Rightarrow p_1 = 1 \Rightarrow p_2 = \left(\frac{\bar{x}_1^1}{\bar{x}_2^2}\right)^{1-\alpha}$

if $0 < \alpha < 1$, $\bar{x}_2^2 \uparrow \Rightarrow p_2 \downarrow$

if $\alpha > 1$ $\bar{x}_2^2 \uparrow \Rightarrow p_2 \uparrow$.

5. a) We can solve it as decision problems.

x : the number that Mr. wants to donate.

y : the number that Ms. wants to donate.

Mr.'s decision problem:

$$\max_{x} U = (14-x)^2 \cdot (14-y) \cdot (8+y+x)$$

$$\text{s.t.} \quad 0 \le x \le 14.$$

$$\text{FOC: } \frac{\partial U}{\partial x} = -2(14-x)(14-y)(8+y+x) + (14-x)^2(14-y)$$
$$= (14-x)(14-y) \cdot (-2-3x-2y) \le 0 \quad \forall\ 0 \le x \le 1$$

Note: only $\frac{\partial U}{\partial x} > 0$, the Mr. would want to denote, because the marginal utility would be larger than c

in this problem, $\forall\ 0 \le x \le 14$, $\frac{\partial U}{\partial x} \le 0$,

so Mr. would do not donate.

Ms.'s decision problem is symmetric, so she would not donate.

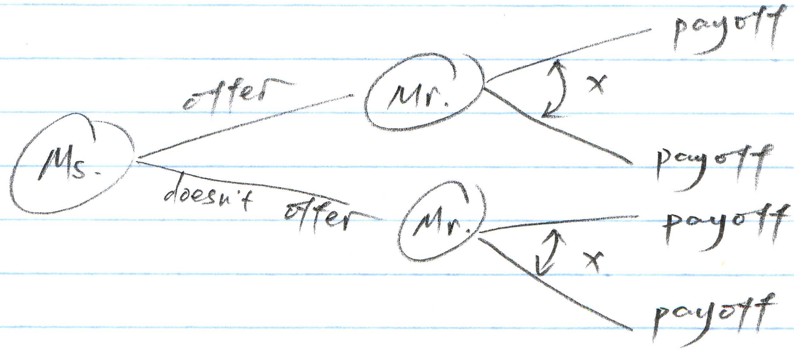
(b). before donation, utilities are: After donation:

	before		after
for Mr.	$(14)^2 \times 14 \times 8 = 21952$	$<$	$(13)^2 \times 13 \times 10 = 21970$
Ms.	$14 \times (14)^2 \times 8 = 21952$	$<$	$13 \times (13)^2 \times 10 = 21970$
student.	$14 \times 14 \times 8^2 = 12544$	$<$	$13 \times 13 \times 10^2 = 16900$

So, if Mr. and Ms. give a crystal to the student, everyone would be better off.

This implies, CE does not have to be PO in each case.

(c). We can understand the problem as a game follows:



Mr.'s decision:

$$\max_x \quad u = (14-x)^2 \cdot (14-x)^2 \cdot (8+2x)$$

$$\text{s.t.} \quad 0 \le x \le 14$$

FOC: $$\frac{\partial u}{\partial x} = -3(14-x)^2(8+2x) + 2(14-x)^3$$

$$= (14-x)^2 \cdot (4-8x)$$

$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 4 \times 14^2 > 0$ Mr. would donate the first unit

$\left.\frac{\partial u}{\partial x}\right|_{x=1} = -4 \times 13^2 < 0$ Mr. would not donate the second unit.

We also know that Ms.'s utility:

$13 \times (13)^2 \times 10 > 14 \times 14^2 \times 8$

So, Ms. would make the offer and match the donation