

Copyright (C) 2006 David K. Levine

This document is an open textbook; you can redistribute it and/or modify it under the terms of version 1 of the open text license amendment to version 2 of the GNU General Public License. The open text license amendment is published by Michele Boldrin et al at <http://levine.sscnet.ucla.edu/general/gpl.htm>; the GPL is published by the Free Software Foundation at <http://www.gnu.org/copyleft/gpl.html>.

Discrete Dynamic Programming

$\alpha \in A$ action space: compact subset of finite dimensional space

$y \in Y$ state space: countable

$\pi(y'|y, \alpha)$ transition probability, continuous in α

$$S(y) \equiv \{y' | \exists \alpha \pi(y'|y, \alpha) > 0\}$$

$S(y)$ is finite

CASE 1: Y, A are finite

CASE 2: Y is a tree; only immediate successors have positive probability

period utility $u(\alpha, y)$ with discount factor $0 \leq \delta < 1$

Definitions

finite histories $h = (y_1, y_2, \dots, y_t)$ with $t(h) = t$ $y(h) = y_t; h-1; y_1(h);$
 $h' \geq h$

a history is feasible if $y_\tau \in \mathcal{S}(y_{\tau-1})$

H space of all feasible finite histories;

this is countable

strategies $\sigma: H \rightarrow A$

Σ space of all strategies

strong Markov strategy $\sigma(h) = \sigma(h')$ if $y(h) = y(h')$

a strong Markov strategy is equivalent to a map

$$\sigma: Y \rightarrow A$$

$$\pi(h|y_1, \sigma) \equiv$$

$$\begin{cases} \pi(y(h)|y(h-1), \sigma(h-1))\pi(h-1|y_1, \sigma) & t(h) > 1 \\ 1 & t(h) = 1 \text{ and } y_1(h) = y_1 \\ 0 & t(h) = 1 \text{ and } y_1(h) \neq y_1 \end{cases}$$

$$V(y_1, \sigma) \equiv (1 - \delta) \sum_{h \in H} \delta^{t(h)-1} u(\sigma(h), y(h)) \pi(h|y_1, \sigma)$$

u is bounded by \bar{u} and continuous in α

Dynamic Programming Problem

(*) maximize $V(y_1, \sigma)$ subject to $\sigma \in \Sigma$

a value function is a map $v: Y \rightarrow \mathfrak{R}$ also bounded by \bar{u}

two infinite dimensional vectors: strategies and value functions

$\mathfrak{R}^\infty, l_\infty$

Lemma 1: a solution to (*) exists

Definition: the value function

$$v(y_1) \equiv \max_{\sigma \in \Sigma} V(y_1, \sigma)$$

Bellman equation

we define a map $T : \ell_\infty \rightarrow \ell_\infty$ by $w' = T(w)$ if

$$w'(y_1) = \max_{\alpha \in A} (1 - \delta)u(\alpha, y_1) + \delta \sum_{y'_1 \in \mathcal{S}(y_1)} \pi(y'_1 | y_1, \alpha) w(y'_1)$$

Lemma 2: the value function is a fixed point of the Bellman equation

$$T(v) = v$$

Lemma 3: the Bellman equation is a contraction mapping

$$\|T(w) - T(w')\| \leq \delta \|w - w'\|$$

Corollary: the Bellman equation has a unique solution

Conclusion 1: the unique solution to the Bellman equation is the value function

Lemma 4: there is a strong Markov optimum that may be found from the Bellman equation

proof: define the strong Markov plan, show that it yields a present value equal to the value function

$$v(y_1(h)) = (1 - \delta) \sum_{t(h) < T} \delta^{t(h)-1} \pi(y(h)|y_1(h), \sigma) u(\sigma(h), y(h)) + (1 - \delta) \sum_{t(h) = T} \delta^T \pi(y(h)|y_1(h), \sigma) v(h)$$