Discrete Dynamic Programming

$\alpha \in A$ action space: compact subset of finite dimensional space

$y \in Y$ state space: countable

$\pi(y' | y, \alpha)$ transition probability, continuous in α

$$S(y) \equiv \{ y' | \exists \alpha \; \pi(y' | y, \alpha) > 0 \}$$

$S(y)$ is finite

CASE 1: Y, A are finite

CASE 2: Y is a tree; only immediate successors have positive probability

period utility $u(\alpha, y)$ with discount factor $0 \le \delta < 1$

Definitions

finite histories $h = (y_1, y_2, \ldots, y_t)$ with $t(h) = t$ $y(h) = y_t$; $h - 1$; $y_1(h)$; $h' \geq h$

a history is feasible if $y_\tau \in S(y_{\tau-1})$

H space of all feasible finite histories; this is countable

strategies $\sigma: H \to A$

Σ space of all strategies

strong Markov strategy $\sigma(h) = \sigma(h')$ if $y(h) = y(h')$

a strong Markov strategy is equivalent to a map

$$\sigma : Y \rightarrow A$$

$$\pi(h|y_1,\sigma) \equiv$$

$$\begin{cases} \pi(y(h)|y(h-1),\sigma(h-1))\pi(h-1|y_1,\sigma) & t(h)>1 \\ 1 & t(h)=1 \text{ and } y_1(h)=y_1 \\ 0 & t(h)=1 \text{ and } y_1(h)\neq y_1 \end{cases}$$

$$V(y_1,\sigma) \equiv (1-\delta)\sum_{h\in H} \delta^{t(h)-1} u(\sigma(h), y(h))\pi(h|y_1,\sigma)$$

u is bounded by $\overline{u}$ and continuous in α

Dynamic Programming Problem

(*) maximize $V(y_1, \sigma)$ subject to $\sigma \in \Sigma$

a value function is a map $v: Y \rightarrow \Re$ also bounded by $\bar{u}$

two infinite dimensional vectors: strategies and value functions

$\Re^\infty, \ell_\infty$

Lemma 1: a solution to (*) exists

Definition: the value function

$$v(y_1) \equiv \max_{\sigma \in \Sigma} V(y_1, \sigma)$$

Bellman equation

we define a map $T : \ell_\infty \to \ell_\infty$ by $w' = T(w)$ if

$$w'(y_1) = \max_{\alpha \in A}(1-\delta)u(\alpha, y_1) + \delta \sum_{y'_1 \in S(y_1)} \pi(y'_1 | y_1, \alpha) w(y'_1)$$

Lemma 2: the value function is a fixed point of the Bellman equation $T(v) = v$

Lemma 3: the Bellman equation is a contraction mapping

$$\|T(w) - T(w')\| \leq \delta \|w - w'\|$$

Corollary: the Bellman equation has a unique solution

Conclusion 1: the unique solution to the Bellman equation is the value function

Lemma 4: there is a strong Markov optimum that may be found from the Bellman equation

proof: define the strong Markov plan, show that it yields a present value equal to the value function

$$
\begin{aligned}
&v(y_1(h)) = \\
&(1-\delta)\sum_{t(h)<T} \delta^{t(h)-1}\pi(y(h)|y_1(h),\sigma)u(\sigma(h),y(h)) + \\
&(1-\delta)\sum_{t(h)=T} \delta^{T}\pi(y(h)|y_1(h),\sigma)v(h)
\end{aligned}
$$