Notes on Period Length in LR-SR Games

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payoff matrix

	Player 2		
Player 1		L	R
	+1	$-x$,0	1,1
	-1	$-x$,0	2,-1

player 2 plays **L** in every Nash equilibrium

player 1's static Nash equilibrium payoff is $-x$

also the minmax payoff for player 1

end of each stage game, public signal $z \in \mathbb{R}$ depends only on action taken by player 1; probability distribution of the public signal is $F(z \mid a_1)$

F either differentiable and strictly increasing, or corresponds to a discrete random variable

$f(z \mid a_1)$ denote the density function

monotone likelihood ratio condition $f(z \mid a_1 = +1)/\, f(z \mid a_1 = -1)$ non-decreasing in z

player 2's action is publicly observed

availability of a public randomization device

τ is length of the period

τ player 1 a long-run player with discount factor $\delta = 1 - r\tau$

faces infinite series of short-run opponents

most favorable perfect public equilibrium for LR characterized by the largest value v that satisfies the incentive constraints

$$v = (1-\delta)1 + \delta \int w(z) f(z \mid a_1 = +1) dz$$

$$v = (1-\delta)2 + \delta \int w(z) f(z \mid a_1 = -1) dz$$

$$v \geq w(z) \geq -x$$

or $v = 0$ if no solution exists

second incentive constraint must hold with equality, since otherwise it would be possible to increase the punishment payoff w while maintaining incentive compatibility from the monotone likelihood ratio condition, iif there is a solution, $w(z)$ it must have cut-point form

$$w(z) = \begin{cases} v & z \geq z* \\ w* & z < z* \end{cases}.$$

Proof:

Any w that is part of a solution to maximizing v subject to the incentive constraints must be a solution, for some constant c to

$\max \int w(z) f(z \mid a_1 = +1) dz$ subject to $\max \int w(z) f(z \mid a_1 = -1) dz \leq c$

Let λ be the Lagrange multiplier for this constraint; the first order condition is $f(z \mid a_1 = +1) = \lambda f(z \mid a_1 = -1)$, or

$$\frac{f(z \mid a_1 = +1)}{f(z \mid a_1 = -1)} = \lambda$$

The monotone likelihood ratio condition implies any solution to this has the cutoff form.

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Define

$$p = \int_{-\infty}^{z^*} f(z \mid a_1 = +1)dz, q = \int_{-\infty}^{z^*} f(z \mid a_1 = -1)dz.$$

enables us to characterize most favorable perfect public equilibrium for LR as largest value v that satisfies incentive constraints

$$v = (1-\delta)1 + \delta(v - p(v-w))$$

$$v = (1-\delta)2 + \delta(v - q(v-w))$$

$$v \geq w \geq -x$$

or $v = 0$ if no solution exists.

linear programming problem characterizing the most favorable equilibrium has solution if and only if

$$(*) \quad r \leq \frac{1}{\tau + ((1+x)[(q-p)/\tau] - [p/\tau])^{-1}}$$

in which case the solution is given by

(**) $$v^* = 1 - p/(q - p).$$

The Poisson Case [Abreu, Milgrom and Pearce, 1991]

signal generated by underlying Poisson process in continuous time

Poisson arrival rate λ_p if the action taken by LR is **+1** and $\lambda_q > \lambda_p$ if the action taken by LR is **–1**

Poisson signal is being “bad news”

the number of signals received during the previous interval of length τ. cutoff point is how many “bad signals” received before punishment $v - w$ is triggered

period length τ is short, can ignore possibility of more than one signal, so the cutoff is necessarily a single signal

approximation: the probability triggering punishment $p = \lambda_p \tau, q = \lambda_q \tau$.

in (**)

$$v^* = \frac{\lambda_p}{\lambda_q - \lambda_p}.$$

in (*)

$$r \leq (1 + x)(\lambda_q - \lambda_p) - \lambda_p.$$

independent of the period length

if satisfied, best equilibrium payoff is v^*, independent of period length and worst possible punishment x.

The Diffusion Case [Faingold and Sanikov [2005]

signaled generated by underlying diffusion process in continuous time

drift in process given by the long-run player's action ($a_1 =$+1 or –1) and the instantaneous variance of the diffusion σ^2

random variable z is increment to the diffusion process during the previous interval of length τ

if the long-run player's action is a_1 signal is normally distributed with mean $a_1\tau$ and variance $\sigma^2\tau$.

allow the variance of the signal z to be given by $\sigma^2\tau^{2\alpha}$ where $\alpha < 1$

diffusion case corresponds to $\alpha = 1/2$

if Φ is the standard normal cumulative distribution

$$p = \Phi\left(\frac{z * -\tau}{\sigma\tau^{\alpha}}\right)$$

$$q = \Phi\left(\frac{z * +\tau}{\sigma\tau^{\alpha}}\right)$$

fix x and consider best equilibrium for player 1 as $\tau \rightarrow 0$

Proposition 1: For any $\alpha < 1$ there exists $\underline{\tau} > 0$ such that for $0 < \tau < \underline{\tau}$ (*) has no solution.

Proof:

$z*(\tau)$ cutoff when period length is τ

work with normalized cutoff

$$\zeta(\tau) = \frac{\tau - z*(\tau)}{\sigma\tau^{\alpha}}$$

So:

$$p = \Phi(-\zeta(\tau))$$

$$q = \Phi\left(\frac{2\tau^{1-\alpha}}{\sigma} - \zeta(\tau)\right).$$

if not (*) not violate then $(q-p)/\tau$ and p/τ both remain bounded away from zero as $\tau \to 0$

show that if $(q-p)/\tau$ remains bounded away from zero, then necessarily $p/\tau \to \infty$

compute $(q-p)/\tau$ using the mean value theorem to observe that for each τ there is a number $f(\tau)$, $0 \leq f(\tau) \leq 1$, such that

$$\begin{aligned}\frac{q-p}{\tau} &= \frac{\Phi\left(\dfrac{2\tau^{1-\alpha}}{\sigma} - \zeta(\tau)\right) - \Phi(-\zeta(\tau))}{\tau} \\ &= \left(\frac{1}{\sigma\tau^{\alpha}}\right)\phi\left(-\zeta(\tau) + f(\tau)\frac{2\tau^{1-\alpha}}{\sigma}\right)\end{aligned}$$

Let $c(\tau) = (q-p)/\tau$, we can invert this relationship to find

$$\phi\left(-\zeta(\tau) + f(\tau)\frac{2\tau^{1-\alpha}}{\sigma}\right) = c(\tau)\sigma\tau^{\alpha}$$

$$\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}\left(-\zeta(\tau) + f(\tau)\frac{2\tau^{1-\alpha}}{\sigma}\right)^2) = c(\tau)\sigma\tau^{\alpha}$$

(***)

$$\left(-\zeta(\tau) + f(\tau)\frac{2\tau^{1-\alpha}}{\sigma}\right)^2 = -2\log(\sqrt{2\pi}c(\tau)\sigma) - 2\alpha\log(\tau)$$

$$\zeta(\tau) = -\sqrt{-2\log(\sqrt{2\pi}c(\tau)\sigma) - 2\alpha\log(\tau)} + f(\tau)\frac{2\tau^{1-\alpha}}{\sigma}$$

want to show $p\,/\,\tau \to \infty$

$c(\tau)$ is bounded away from zero so

$\zeta(\tau) \le \sqrt{-\log b - 2\alpha \log \tau} + a\tau^{1-\alpha}$

and

$p\,/\,\tau \ge \Phi(-\sqrt{-\log b - 2\alpha \log \tau} - a\tau^{1-\alpha})\,/\,\tau$

by mean value theorem for some $\tau' \in [0, \tau]$

$$\Phi(-\sqrt{-\log b - 2\alpha\log\tau} - a\tau^{1-\alpha})/\tau$$

$$= \left(\frac{\alpha}{\tau'\sqrt{-\log b - 2\alpha\log\tau'})} - a(1-\alpha)(\tau')^{-\alpha}\right)\phi(-\sqrt{-\log b - 2\alpha\log\tau'} - a(\tau')^{1-\alpha})$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{\alpha}{\tau'\sqrt{-\log b - 2\alpha\log\tau'})} - a(1-\alpha)(\tau')^{-\alpha}\right)$$

$$\times\exp(-1/2)\left(-\log b - 2\alpha\log\tau' + 2a(\tau')^{1-\alpha}\sqrt{-\log b - 2\alpha\log\tau'} + a^2(\tau')^{2-2\alpha}\right)$$

$$= \frac{b^{1/2}}{\sqrt{2\pi}}\left(\frac{\alpha}{\tau'\sqrt{-\log b - 2\alpha\log\tau'})} - a(1-\alpha)(\tau')^{-\alpha}\right)$$

$$\times(\tau')^{\alpha}\exp(-a(\tau')^{1-\alpha}\sqrt{-\log b - 2\alpha\log\tau'})\exp(-1/2)\left(a^2(\tau')^{2-2\alpha}\right)$$

$$\to \frac{\alpha b^{1/2}}{\sqrt{2\pi}}\left(\frac{(\tau')^{\alpha-1}}{\sqrt{-2\alpha\log\tau'})}\right)\exp(-a\frac{\sqrt{-2\alpha\log\tau'}}{(\tau')^{\alpha-1}})$$

apply L'Hopital's rule to show that

$$\frac{\tau^{\alpha-1}}{\sqrt{-\log\tau}} \to (1-\alpha)\frac{\tau^{\alpha-2}}{1/\tau} = 1-\alpha)\tau^{\alpha-1} \to \infty.$$

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recall that $v^* = 1 - p/(q-p)$

consider for fixed τ taking very small cutoff $z^* \to -\infty$

as in Mirlees [????] causes likelihood ratio $q/p \to \infty$ so $p/(q-p) \to 0$; that is $v^* \to 1$, the pure commitment value, and the best PPE payoff for the most the long-run player can get when there is no noise in the signal

this solution has $p, q \to 0$, which means that for fixed x and τ and z^* sufficiently negative, (*) must be violated

however, for any choice of z^*, r, τ, there is always an x sufficiently large that (*) holds

the diffusion case the worst punishment determines the best equilibrium

going far enough into the tail of the normal, arbitrarily reliable information is available and can be used to create incentives, provided sufficiently harsh punishment is available