ECON 201B - Game Theory

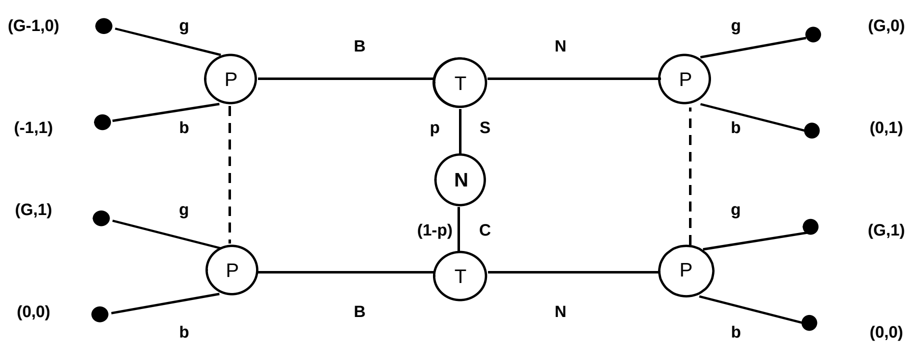
Suggested Answers - Midterm 2

February 23, 2006

1 First Problem

President of the U.S. Bert C. Tree will shortly negotiating an agreement with the Premier of Hapistan. The Premier of Hapistan believes that President Tree is one of two types: sane (S) or crazy (C). The probability of type S is p. Prior to the negotiation, President Tree may bomb another country. The cost to the President of the bombing is zero for a crazy type and one for a sane type. After the bombing, the Premier may offer the President either a good deal (worth $G > 0$ to the President) or a bad deal (worth nothing to the President). The Premier gets one for offering the crazy type the good deal, one for offering the sane type the bad deal, and zero otherwise.

a) Draw the extensive form.



where T represent the President Tree and P the Premier of Hapistan. B is bombing a city, N is not bombing, g is a good deal offered by the Premier and b is a bad deal.

b) Find the pure strategy Nash equilibria of the game for the different values of p and G.

To find Nash equilibria we need to construct the normal form game.

President Tree (T) has the set of strategies: $\{S, C\} \to \{B, N\}$ being $\{S, C\}$ T's types. Premier (P) has the set of strategies: $\{B, N\} \to \{g, b\}$.

	gg	gb	bg	bb
BB	$G-p$ $1-p$	$\underline{\mathbf{G-p}}$ $1-p$	$-p$ $\underline{\mathbf{p}}$	$-p$ $\underline{\mathbf{p}}$
BN	$G-p$ $1-p$	$pG-p$ 0	$G(1-p)-p$ $\underline{\mathbf{1}}$	$-p$ p
NB	$\underline{\mathbf{G}}$ $1-p$	$G(1-p)$ $\underline{\mathbf{1}}$	pG 0	$\underline{\mathbf{0}}$ p
NN	$\underline{\mathbf{G}}$ $1-p$	0 $\underline{\mathbf{p}}$	$\underline{\mathbf{G}}$ $1-p$	$\underline{\mathbf{0}}$ $\underline{\mathbf{p}}$

There are two important cutoffs in the parameters to obtain best responses in pure strategies. For the Premier ($p = \frac{1}{2}$) and for the President Tree ($G = 1$).

For example, in the matrix we denote in bold and underlined the best responses for the case $p > \frac{1}{2}$ and $G > 1$. In this case there is just one pure strategy NE, $\{NN; bb\}$.

Pure strategy Nash equilibria

Doing the same for other cases, pure strategy NE are,

- $p > \frac{1}{2}; G > 1 \Longrightarrow NE = \{(NN, bb)\}$
- $p > \frac{1}{2}; G \leq 1 \Longrightarrow NE = \{(NN, bb); (NB, gb)\}$
- $p < \frac{1}{2}; G > 1 \Longrightarrow NE = \{(NN, gg); (BB, gb); (NN, bg)\}$
- $p < \frac{1}{2}; G = 1 \Longrightarrow NE = \{(NN, gg); (BB, gb); (NN, bg); (NB, gb)\}$
- $p < \frac{1}{2}; G < 1 \Longrightarrow NE = \{(NN, gg); (NN, bg); (NB, gb)\}$

- $p = \frac{1}{2}; G > 1 \Longrightarrow NE = \{(NN, gg); (BB, gb); (NN, bg); (NN, bb)\}$
- $p = \frac{1}{2}; G = 1 \Longrightarrow NE = \{(NN, gg); (BB, gb); (NN, bg); (NN, bb); (NB, gb)\}$
- $p = \frac{1}{2}; G < 1 \Longrightarrow NE = \{(NN, gg); (NN, bg); (NN, bb); (NB, gb)\}$

c) Find a mixed strategy Nash equilibrium of the game for a particular value of p and G.

Many answers (related with the many existing mixed strategy NE) can be proposed.

Let me suggest the case $p = \frac{3}{4}$ and $G = 2$. (In point d) it will become clear why I'm choosing particularly this one). The matrix will look like, (x_i and y_i are the randomization probabilities for T and P respectively).

		y_1	y_2	y_3	y_4
		gg	gb	bg	bb
x_1	BB	$\frac{5}{4}, \frac{1}{4}$	$\frac{5}{4}, \frac{1}{4}$	$-\frac{3}{4}, \frac{3}{4}$	$-\frac{3}{4}, \frac{3}{4}$
x_2	BN	$\frac{5}{4}, \frac{1}{4}$	$\frac{3}{4}, 0$	$-\frac{1}{4}, 1$	$-\frac{3}{4}, \frac{3}{4}$
x_3	NB	$2, \frac{1}{4}$	$\frac{1}{2}, 1$	$\frac{3}{2}, 0$	$0, \frac{3}{4}$
x_4	NN	$2, \frac{1}{4}$	$0, \frac{3}{4}$	$2, \frac{1}{4}$	$0, \frac{3}{4}$

Now, let me propose a randomization by the President Tree between BB and NB (i.e. $x_2 = x_4 = 0$) and a randomization by the Premier between gb and bb (i.e. $y_1 = y_3 = 0$). T is indifferent between BB and NB if $y_2 = \frac{1}{2}$. Likewise P is indifferent between gb and bb if $x_1 = \frac{1}{3}$.

It's easy to check that given this mixing by T, P optimally randomizes, getting an expected payoff of $\frac{3}{4}$ against a payoff of $\frac{1}{4}$ from playing gg or bg. The same is true for T, who gets an expected payoff of $\frac{1}{4}$ against 0 from playing NN or BN.

Hence, this mixed strategy NE is given by $\{\frac{1}{3}BB + \frac{2}{3}NB; \frac{1}{2}gb + \frac{1}{2}bb\}$.

d) From the cases above, give examples of a pooling, separating and hybrid equilibrium?

At this point we will take a couple of NE found above and we will determine if they are sequential equilibria and under which beliefs this happens.

(This is specifically what the question asks you to do. However you can find at the end of this answer key a more comprehensive analysis for all the pure strategy NE discussed in b).

- **Consider the pure strategy NE $(\mathbf{NB}, \mathbf{gb})$.** This can be read as:

  - T playing $S \rightarrow N$ and $C \rightarrow B$.

  - P playing $B \rightarrow g$ and $N \rightarrow b$.

The Premier assesments in this case will be $\mu(S|B) = 0$ and $\mu(S|N) = 1$. His best response will be $B \rightarrow g$ and $N \rightarrow b$. Playing as proposed is optimal for the President Tree for any p as long as $G \leq 1$.

Hence, the NE $(\mathbf{NB}, \mathbf{gb})$ is a separating equilibrium when beliefs are $\boldsymbol{\mu}(\mathbf{S}|\mathbf{B}) = \mathbf{0}$ and $\boldsymbol{\mu}(\mathbf{S}|\mathbf{N}) = \mathbf{1}$

- **Consider the pure strategy NE $(\mathbf{BB}, \mathbf{gb})$.** This can be read as:

  - T playing $S \rightarrow B$ and $C \rightarrow B$.

  - P playing $B \rightarrow g$ and $N \rightarrow b$.

The Premier assesments in this case will be $\mu(S|B) = p$, but it's not clear what $\mu(S|N)$ is. His best response will be $B \rightarrow g$ and $N \rightarrow b$ only if $p \leq \frac{1}{2}$ and $\mu(S|N) \geq \frac{1}{2}$. Playing as proposed is optimal for the President Tree as long as $G \geq 1$. **Hence, the NE (BB, gb) is a pooling equilibrium when beliefs are $\boldsymbol{\mu}(\mathbf{S}|\mathbf{B}) = \mathbf{p}$ and $\boldsymbol{\mu}(\mathbf{S}|\mathbf{N}) \geq \frac{1}{2}$**

- **Consider the mixed strategy NE** (from c) for the case when $p = \frac{3}{4}$ and $G = 2$. $\{\frac{1}{3}BB + \frac{2}{3}NB; \frac{1}{2}gb + \frac{1}{2}bb\}$. This can be read as:

  - T playing $S \rightarrow \frac{1}{3}B + \frac{2}{3}N$ and $C \rightarrow B$.

  - P playing $B \rightarrow \frac{1}{2}g + \frac{1}{2}b$ and $N \rightarrow b$.

Let's check if this is a semi-separating (or hybrid) equilibrium. The information set followed by N is reached only by type S who randomizes, never by type C. Hence $\mu(S|N) = 1$ and P will play $N \longrightarrow b$.

The President Tree of type S is in fact indifferent between playing B and N if $\Pr(g|B) = \frac{1}{2}$. Hence, P should randomize after observing B, which occurs only if $\mu(S|B) = \frac{1}{2}$.

Following a Bayesian rule, $\mu(S|B) = \frac{\Pr(B|S)\Pr(S)}{\Pr(B|S)\Pr(S)+\Pr(B|C)\Pr(C)} = \frac{1}{2}$. Hence $\Pr(B|S) = \frac{1}{3}$

Finally, C would optimally play B since the expected payoff is $\frac{1}{2}$ against the payoff of 0 from playing N. **Hence, this is a hybrid sequential equilibrium when beliefs are $\boldsymbol{\mu}(\mathbf{S}|\mathbf{N}) = \mathbf{1}$ and $\boldsymbol{\mu}(\mathbf{S}|\mathbf{B}) = \frac{1}{2}$**

2 Second Problem

An automobile driver faces three possible outcomes: no accident; medium accident and severe accident. There are two equally probable types of driver, all have a 50% chance of no accident. The high risk driver has a 50% chance of a severe accident and the low risk driver has a 50% chance of a medium accident. The driver type is known only to the driver. The income of a driver is equal to her income of 100 minus the cost of the accident. The utility of a driver with income c is $u(c) = c - c^2/200$.

The cost of an accident is zero for no accident; ten for a medium accident; twenty for a severe accident. You may offer four kinds of contracts: no contract; a payment of ten for a medium accident; a payment of twenty for a severe accident; a payment of ten for a medium accident and a payment of twenty for a severe accident. Which contracts should you offer and what should you charge for each one?

This is a risk averse driver since the second derivative of the utility function is $u''(c) = -\frac{1}{100} < 0$. High risk drivers have zero probability of having medium accidents and low risk drivers never experience severe accidents. We, the insurance company, can condition the payment on the type of accident (known) but not on the type of driver (unknown).

Possible contracts we can offer are,

Case 1: NO contract

Profits in this case are $\boldsymbol{\pi}_1 = \mathbf{0}$

Case 2: Payment of 10 for a medium accident

This contract will be acquired only by low risk drivers since high risk ones never have medium accidents.

The price for the insurance will be determined by making low risk drivers indifferent between buying the policy or not.

Without insurance, $Eu_L(w/o) = \frac{1}{2}u(100) + \frac{1}{2}u(90) = 49.75$

With insurance, $Eu_L(w) = u(100 - p_L)$

being p_L the price for this insurance policy. Equalizing both, $u(100 - p_L) = 49.75$, which implies $\mathbf{p}_L = \mathbf{7.07}$

Profits in this case are, $\pi_2 = \frac{1}{2}\left[\frac{1}{2}(7.07) + \frac{1}{2}(7.07 - 10)\right] = \mathbf{1.035}$ (since the offered contract is accepted just with a probability $\frac{1}{2}$, when you meet a low risk driver).

Case 3: Payment of 20 for a several accident

This contract will be acquired only by high risk drivers since low risk ones never have severe accidents.

Again, the price for the insurance is determined by making high risk drivers indifferent between buying the policy or not.

Without insurance, $Eu_H(w/o) = \frac{1}{2}u(100) + \frac{1}{2}u(80) = 49$

With insurance, $Eu_H(w) = u(100 - p_H)$

being p_H the price for this insurance policy. Equalizing both, $u(100 - p_H) = 49$, which implies $\mathbf{p}_H = \mathbf{14.14}$

Profits in this case are, $\pi_2 = \frac{1}{2}\left[\frac{1}{2}(14.14) + \frac{1}{2}(14.14 - 20)\right] = \mathbf{2.07}$ (since the offered contract is accepted just with a probability $\frac{1}{2}$, when you meet a high risk driver).

Case 4: Payment of 10 for a medium accident AND 20 for a several accident

In this case both types of drivers would be willing to buy the insurance coverage depending on the price. Following the same logic than before,

Low risk drivers will buy the policy only if $p \leq p_L = 7.07$

High risk drivers will buy the policy only if $p \leq p_H = 14.14$

- If $p > 14.14$, no driver would buy the insurance, then $\pi_4 = 0$

- If $7.07 < p \leq 14.14$, only high risk drivers would buy the insurance. Hence there is no point in setting a price smaller than 14.14. With $p = 14.14$, $\pi_4 = \mathbf{2.07}$ (no medium accident would be effectively paid in this case and the contract is sold only with a probability $\frac{1}{2}$).

- If $p \leq 7.07$, both high and low risk drivers would buy the insurance. Hence there is no point in setting a price lower than 7.07. With $p = 7.07$, $\pi_4 = \frac{1}{2}(7.07) + \frac{1}{4}(7.07 - 10) + \frac{1}{4}(7.07 - 20) = -0.43 < 0$.

Hence, it's better for the insurance company to set a price 14.14, attracting only high risk drivers.

Case 5: Offer two different contracts. One that pays 10 for a medium accident and the other that pays 20 for a several accident.

The first contract should be offered by a price $p_L = 7.07$ and the second by a price $p_H = 14.14$. Offering these two possibilities, low risk drivers will choose the first contract and high risk drivers the second one.

Profits in this case will be, $\pi_5 = \frac{1}{2}[7.07 - 5] + \frac{1}{2}[14.14 - 10] = \mathbf{3.105}$

The best an insurance company can do is to offer a menu of contracts. One that pays 10 for a medium accident and costs 7.07 and the other that pays 20 for a several accident and costs 14.14. In this way the profit is 3.105.

Supplemental Section - Sequential Equilibria

	T's Plan	P's assesments	P's b.r.	Optimal for T?
		Separating Equilibria		
$I)$	$S \longrightarrow B; C \longrightarrow N$	$\mu(S\|B) = 1$ $\mu(S\|N) = 0$	$B \longrightarrow b; N \longrightarrow g$	NO $(S \longrightarrow N)$
$II)$	$S \longrightarrow N; C \longrightarrow B$	$\mu(S\|B) = 0$ $\mu(S\|N) = 1$	$B \longrightarrow g; N \longrightarrow b$	YES (if $G \leq 1$)
		Pooling Equilibria		
$III)$	$S \longrightarrow B; C \longrightarrow B$	$\mu(S\|B) = p$ $q = \mu(S\|N) =?$		
	$i)$	$p < \frac{1}{2}; q > \frac{1}{2}$	$B \longrightarrow g; N \longrightarrow b$	YES (if $G \geq 1$)
	$ii)$	$p < \frac{1}{2}; q < \frac{1}{2}$	$B \longrightarrow g; N \longrightarrow g$	NO $(S \longrightarrow N)$
	$iii)$	$p > \frac{1}{2}; q > \frac{1}{2}$	$B \longrightarrow b; N \longrightarrow b$	NO $(S \longrightarrow N)$
	$iv)$	$p > \frac{1}{2}; q < \frac{1}{2}$	$B \longrightarrow b; N \longrightarrow g$	NO $(S \longrightarrow N)$
$IV)$	$S \longrightarrow N; C \longrightarrow N$	$r = \mu(S\|B) =?$ $\mu(S\|N) = p$		
	$i)$	$p > \frac{1}{2}; r > \frac{1}{2}$	$B \longrightarrow b; N \longrightarrow b$	YES
	$ii)$	$p > \frac{1}{2}; r < \frac{1}{2}$	$B \longrightarrow g; N \longrightarrow b$	NO $(C \longrightarrow B)$
	$iii)$	$p < \frac{1}{2}; r > \frac{1}{2}$	$B \longrightarrow b; N \longrightarrow g$	YES
	$iv)$	$p < \frac{1}{2}; r < \frac{1}{2}$	$B \longrightarrow g; N \longrightarrow g$	YES

Considering all pure strategy NE from b),

- (NB, gb) is a separating equilibrium (for any p and $G \leq 1$). (in II)
- (BB, gb) is a pooling equilibrium (for $p \leq \frac{1}{2}$, $G \geq 1$ and $\mu(S|N) \geq \frac{1}{2}$). (From $III)i)$)
- (NN, bb) is a pooling equilibrium (for $p \geq \frac{1}{2}$ and $\mu(S|B) \geq \frac{1}{2}$). $(IV)i)$)
- (NN, bg) is a pooling equilibrium (for $p \leq \frac{1}{2}$ and $\mu(S|B) \geq \frac{1}{2}$). (From $IV)iii)$)
- (NN, gg) is a pooling equilibrium (for $p \leq \frac{1}{2}$ and $\mu(S|B) \leq \frac{1}{2}$). (From $IV)iv)$)