CHAPTER II: LONG RUN COLLUSION IN A PARTIALLY MYOPIC INDUSTRY

0. Introduction

Will partially myopic firms collude in the long run? In the first chapter we saw in a quadratic symmetric example the absence of frictional costs implies that they will. This chapter studies an unrestricted technology without frictional costs. I ask: how are long run steady states of the adjustment process related to pareto efficient outcomes of the static game? The answer has two parts:

- In "almost all" games steady states and outcomes satisfying the first order conditions for static pareto efficiency "almost" coincide.

- The general stability analysis is intractible. However, it is possible to show that with identical firms and symmetric initial conditions a steady state is stable if and only if it is (locally) efficient.

The paper has four sections. Section one reviews the model of chapter one. Section two studies steady states. Section three focuses on the stability of steady states. Section four discusses the implication of an approximation introduced in section one. The final section summarizes the conclusions of the paper.

1. Review

There are N firms, entry is prohibited and each firm j controls its own output x^j. The output vector x is presumed to lie in X, an open subset of $\mathbb{R}^N$. By assuming X is open behavior on the boundary is ignored.

Attention is limited to the case in which communication between firms can occur and there are no counter-reactions. Thus, from (1-2) of chapter one the movement of x is given by

$$\dot{x}^j = \sum_{k=1}^{N} R^j_k y^k \tag{1-1}$$

where y^j are the autonomous output controls. From (1-3) of chapter one the reaction coefficients move according to

$$\dot{R}^j_k = S^j_k \tag{1-2}$$

The profits of firm j are given by a smooth function $\pi^j : X \to \mathbb{R}$. Note that there are no frictional costs of reacting. Let $\pi^j_k = \partial\pi^j/\partial x^k$. To insure that firms are able to affect opponent's profits it is assumed that for $x \in X$ and $j \neq k$ $\pi^j_k(x) < 0$. Let $\pi^{\cdot}$ be the vector of profit functions, π_k be the column vector $(\pi^j_k)_{j=1}^N$, π^j the row vector $(\pi^j_k)_{k=1}^N$ and π be the matrix with rows π^j. Since $\mathcal{X}$ is open the static game with profit functions π^j may not have any efficient points. To rule out degeneracy at least some point $x \in \mathcal{X}$ should satisfy the first order conditions for pareto efficiency.

As a second regularity condition it is assumed that for some $x \in \mathcal{X}$ $\det(\pi(x))=0$. In section two it is shown that this is in fact the first order condition for efficiency.

Firm j's intertemporal preferences are described by a discount rate ρ^j. The corresponding discount factor is $\delta^j = 1/\rho^j$.

The behavioral model is the same as in chapter one. As in equation (2-4) of chapter one the strategies $\tilde{f}^j=y^j$ and $\tilde{g}_k^j=s_k^j$ are given by

$$\tilde{f}^j = bk_j^j \sum_{k=1}^{N} \frac{\partial \hat{A}^j}{\partial x^k} R_j^k$$

$$\tilde{g}_k^j = bk_k^j \frac{\partial \hat{A}^j}{\partial R_k^j} \tag{1-3}$$

where the bk_k^j are the partial adjustment coefficients. The approximate present value of income from (2-12) in chapter one is

$$\hat{A}^j = \delta^j \pi^j + (\delta^j)^2 b \sum_{\ell=1}^{N} \pi_\ell^j \sum_{k=1}^{N} R_k^\ell \, k_k^k \delta^k \sum_{m=1}^{N} R_k^m \pi_m^k \tag{1-4}$$

Thus by (1-3) and (1-4)

$$\tilde{f}^j = b\, k_j^j \sum_{\ell=1}^{N} \frac{\partial \hat{A}^j}{\partial x^\ell} R_j^\ell = b\delta^j k_j^j \sum_{\ell=1}^{N} \pi_\ell^j R_j^\ell + O(b^2) \tag{1-5}$$

which as in chapter one simplifies to

$$y^j = \tilde{f}^j \approx \sum_{\ell=1}^{N} \pi_\ell^j R_j^\ell \tag{1-6}$$

where $b\delta^j k^j_j$ is normalized to one by choosing appropriate units for π^j and the term of $O(b^2)$ is dropped. Section four discusses the implications of this approximation in greater detail.

Finally from (1-3) and (1-4) the motion of the reaction coefficients is

$$\dot{R}^j_k = b\,k^j_k\,\partial\hat{A}^j/\partial R^j_k$$

$$= b(\delta^j)^2 k^j_k[\pi^j_j \sum_{m=1}^{N} R^m_k \pi^k_m + \pi^k_j \sum_{m=1}^{N} R^m_k \pi^j_m]$$

$$= b\eta^j_k[\pi^k_j \sum_{m=1}^{N} R^m_k \pi^j_m + \pi^j_j y^k] \qquad (1\text{-}7)$$

where the constant $\eta^j_k = (\delta^j)^2 k^j_k$ and use is made of the normalization rule $b\delta^j k^j_j = 1$.

2. Equilibrium

The remainder of the paper addresses the following question: what is the relationship between stable steady states of the dynamical system described in the first section and pareto efficient outcomes of the game? Mathematically this divides into two issues: how are steady states related to outcomes which satisfy the first order conditions for pareto efficiency (called FOPE), and how are the stability conditions related to the second order conditions for pareto efficiency? This section takes up the first issues and yields the following conclusion: in "almost all" games which satisfy the required regularity conditions steady states and FOPE coincide "amost exactly." The following section examines the relationship between stability and local efficiency.

A point x is supportable as a steady state (or is simply called a steady state) if there is some reaction matrix R such that $\dot{x}(x,R) = 0$. In this case the dynamical system is motionless for all time regardless of whether $y = 0$. There are two possible types of steady states. If $y = 0$ by (1-17) $\dot{x} = 0$. This is called an autonomous steady state to reflect the fact that the autonomous action control variables y are stationary. Steady states at which $y \neq 0$ are called non-autonomous. The first half of the section examines autonomous steady states and shows that they are all FOPE. As a partial converse in "almost all" games the FOPE form an N-1 dimensional set of which at worst an N-2 dimensional subset fail to be autonomous steady states. The second half of the section examines the possibility that some non-autonomous steady states might fail to be FOPE and demonstrates that in "almost all" games such exceptional steady states are a set of isolated points. The section concludes by showing graphically the relationship between FOPE and steady states in the "generic" case.

Autonomous steady states are characterized by the conditions $y = 0$ $\dot{R} = 0$. To analyze the set of points x which are supportable as autonomous steady states R must be eliminated from these equations. The resulting condition will then be contrasted with the first order efficiency conditions to show that autonomous steady states are FOPE. Before proving a partial converse a digression will give a mathematical definition of "almost all" games.

Equating the expressions for y and $\dot{R}$ from (1- 6) and (1- 7) to zero gives the condition for an autonomous steady state

$$\sum_{p=1}^{N} \pi_p^j R_j^p = 0 \qquad j = 1,\ldots, N \qquad (2\text{-}1)$$

$$\pi_k^j \sum_{p=1}^{N} \pi_p^j R_k^p = 0 \qquad j = k = 1,\ldots, N \quad j \neq k \qquad (2\text{-}2)$$

Since the $\pi_k^j \neq 0$ (by assumption) they can be eliminated from (2-2) to yield the equivalent condition

$$\sum_{p=1}^{N} \pi_p^j R_k^p = 0 \qquad j, k = 1,\ldots, N \quad j \neq k \qquad (2\text{-}3)$$

It is instructive to combine the N^2 equations in (2-1) and (2-3) into the N vector equations

$$\pi R_k = 0 \qquad k = 1,\ldots,N \qquad (2\text{-}4)$$

where $\pi=\{\pi_k^j\}$ with j subscripting rows and k subscripting columns and $R_k=(R_k^j)_{j=1}^N$

If it is to be possible to solve (2-4) for N vectors satisfying $R_k^k = 1$ then π must be singular and admit in its null space a vector $\gamma = (\gamma^j)_{j=1}^N$ with $\gamma^j \neq 0$ for any j. I will call such a matrix regular singular. If π is regular singular setting $R_k^j = \gamma^j/\gamma^k$ solves (2-4) and satisfies the restriction $R_k^k = 1$. So it is necessary and sufficient for x to be an autonomous steady state that $\pi(x)$ be regular singular.

How does this compare with the first order condition for pareto efficiency? At a pareto efficient point for a non-zero vector of weights $\mu = (\mu^j)_{j=1}^N$ the weighted sum $\sum_{j=1}^N \mu^j \pi^j$ must be maximized. The first order conditions (which must be satisfied since $\mathcal{X}$ is an open set) are

$$\sum_{j=1}^N \mu^j \pi_k^j = 0 \qquad k = 1,\ldots, N \qquad (2\text{-}5)$$

which in matrix notation is

$$\mu'\pi = 0 \qquad (2\text{-}6)$$

or the condition that π be singular. This condition will be taken as the definition of a FOPE, the restriction that the weights μ^j have the same sign being viewed in this terminology as part of the second order conditions for efficiency. This definition is illustrated graphically at the end of the section. Note incidentally the difference between the weights γ corresponding to reactions and the weights μ corresponding to utility weights: the former satisfies

$\pi\gamma = 0$, the latter $\mu'\pi = 0$.

All autonomous steady states are FOPE. The converse is false since not all singular matrices are regular singular. To proceed further we must digress to give some technical definitions. By an L dimensional set is meant a finite collection of L or lower dimensional manifolds at least one of which has dimension L. For example in R^3 the set consisting of the $x^1 - x^2$ plane and the x^3 axis is a two-dimensional set. In appendix (A) it is shown that the set of singular NxN matrices is an $N^2 - 1$ dimensional set. Since singular matrices π satisfy the single restriction det $(\pi) = 0$ and lie in an N^2 dimensional space this should not be too surprising. A singular matrix π which is not regular singular satisfies the additional restriction that every vector in its null space has $\gamma^j = 0$ for some j. As appendix (A) demonstrates these matrices lie in a $N^2 - 2$ dimensional set.

That most singular matrices are regular singular unfortunately does not imply that in an arbitrary game most FOPE are autonomous steady states. In a linear game $\pi(x)$ does not depend on x and it could well be that π is non-regular singular so that the entire game consists of FOPE which are not autonomous steady states. This example is clearly rather special and it would be useful to assert that this does not happen in "most" games.

It is now necessary to introduce some technical concepts from differential topology. References are Guillemin and Pollack [6], Milnor [15] and Hirsch [7]. The map π is said to meet a submanifold S of a manifold M transversally iff for each x with $\pi(x) \varepsilon S$ the tangent to π at x and the tangent space to S at $\pi(x)$ span the tangent space to M at $\pi(x)$. It meets an L dimensional set S transversally in an N^2 dimensional space M if it meets each component manifold of S transversally. In this case the implicit function theorem shows that the set of x such that $\pi(x) \varepsilon S$ has the "right" dimension--that is, dimension $N - (N^2 - L)$. Thus if π meets both the singular and non-regular singular matrices transversally the FOPE are N - 1 dimensional and the points which are not autonomous steady states are confined to an N - 2 dimensional subset. Note that an L-dimensional set may be empty, although not in the case of FOPE which are assumed to exist.

As the example above shows not all games meet a given set transversally. "Most" games, however, do have this property. Let G be the set of all C^2 mappings $\mathcal{X} \dashrightarrow \mathbb{R}^N$ and let G^R be the mappings $\pi^{\cdot}$ which satisfy the regularity conditions $\pi^j_k(x) < 0$ for any $j \neq k$, $x \varepsilon \mathcal{X}$ and for some $x \varepsilon \mathcal{X}$ $\det(\pi(x)) = 0$. Then G is the set of all games and G^R the set of regular games. A topology can be introduced on G and G^R by specifying the neighborhoods of each game π. Let $K = K_i$ be a covering of $\mathcal{X}$ with compact sets such that no point lies in more than a finite number of these sets and let $\varepsilon = \varepsilon_i$ be positive real numbers. A neighborhood $U(\pi^{\cdot}, K, \varepsilon)$ of $\pi^{\cdot}$ corresponding to K and ε are all games $\hat{\pi}^{\cdot}$ satisfying

$$\sup \left\{ |\hat{\pi}^j(x) - \pi^j(x)|,\ |\hat{\pi}^j_k(x) - \pi^j_k(x)|,\ |\hat{\pi}^j_{k\ell}(x) - \pi^j_{k\ell}(x)| \right\} \leq \varepsilon_i$$

$$x \varepsilon K_i;\ j, k, \ell = 1, \ldots, N \tag{2-7}$$

that is $\hat{\pi}^{\cdot}$ must be close to $\pi^{\cdot}$ in the assigned payoff and its first two derivatives. This is known as the Whitney C^2 topology.

A residual subset in G is the countable intersection of dense open sets. In $\mathbb{R}$ an example of a residual set is the set of irrational numbers--they are the intersection of the sets consisting of the real line with one rational point deleted. It can be shown that any residual subset of G is dense, and by almost all games in G is meant all games in some residual subset.

A theorem known as the jet transversality theorem asserts that almost all maps $\pi^{\cdot}\varepsilon G$ have derivative maps π that meet a given set transversally. This paper examines only regular games and requires a definition of residual in G^R rather than G. In appendix (B) it is shown that if D is dense in G then $D \cap G^R$ is dense in G^R. In light of this a subset of G^R is called residual iff it is the intersection of a residual set in G with G^R. This gives the desired conclusion that almost all games in G^R (and in particular all games in a dense subset) have derivative maps which meet a given set transversally.

The preceding discussion is conveniently summarized in two propositions.

Proposition (2-1): If S is an L dimensional set then for almost all games the set of $x \varepsilon \mathcal{X}$ which satisfy the restriction $\pi(x) \varepsilon S$ is $N - (N^2 - L)$ dimensional.

Proposition (2-2): All autonomous steady states are FOPE. In almost all games FOPE are N - 1 dimensional and are all autonomous steady states except possibly in an N - 2 dimensional set.

Exceptional Steady States: At an autonomous steady state y = 0, and the first order conditions for pareto efficiency are satisfied. It is possible, however that $\dot{x} = 0$ and $y \neq 0$ in which case the preceding analysis does not apply. Here it is shown that in almost all games there are at most an isolated set of points which are steady states but not FOPE There are two steps in this undertaking: first R is eliminated from the equations $\dot{x} = 0$ $\dot{R} = 0$; then an application of Proposition (2-1) gives the desired genericity result.

The conditions for a (not necessarily autonomous) steady state are given by equating the expressions for $\dot{x}$ and $\dot{R}$ from (1-1) and (1- 7) to zero

$$\sum_{k=1}^{N} R_k^j y^k = 0 \qquad j = 1,\ldots, N \tag{2-8}$$

$$[\pi_k^j(\sum_{p=1}^{N} \pi_p^j R_k^p) + \pi_j^j y^k] = 0 \qquad j \neq k \tag{2-9}$$

To eliminate R from these equations use $\pi_k^j \neq 0$ $j \neq k$ to define

$$\lambda_k^j = \begin{cases} 1 & j = k \\ -\pi_j^j/\pi_k^j & j \neq k \end{cases} \tag{2-10}$$

Then (2-9) is equivalent to

$$\sum_{p=1}^{N} \pi_p^j R_k^p = y^k \lambda_k^j \qquad j \neq k \tag{2-11}$$

The equation (1- 6) defining y^k is

$$y^k = \sum_{p=1}^{N} \pi_p^k . R_k^p \tag{2-12}$$

which can be expressed as

$$\sum_{p=1}^{N} \pi_p^k R_k^p = y^k \lambda_k^k \tag{2-13}$$

since $\lambda_k^k = 1$ by definition. The N^2 equations (2-11) and (2-13) can be combined into the N vector equations

$$\pi R_k = y^k \lambda_k \tag{2-14}$$

Throughout the remainder of this discussion it is assumed that the steady state is not a FOPE. In this case π is non-singular and (2-14) can be solved for the reaction coefficients.

$$R_k = y^k \pi^{-1} \lambda_k \tag{2-15}$$

The equation for R_k^k (which is one by definition) is

$$1 = R_k^k = y^k (\pi^{-1})^k \lambda_k \tag{2-16}$$

from which the autonomous controls are

$$y^k = 1/(\pi^{-1})^k \lambda_k \tag{2-17}$$

Observe that this implies $y^k \neq 0$ for any k, which is consistent with the earlier finding that the steady state must be non-autonomous since it is not a FOPE.

The N vector equations (2-15) can be rewritten using (2-17) as the N^2 scalar equations

$$R_k^j = (\pi^{-1})^j \lambda_k / [(\pi^{-1})^k \lambda_k] \tag{2-18}$$

So far the first steady state condition $\dot{x} = 0$ given in (2-8) has not been used. To eliminate R from the steady state conditions (2-8) and (2-9) substitute (2-18) into (2-8) to get the necessary conditions

$$\sum_{k=1}^{N} (\pi^{-1})^j \lambda_k/[(\pi^{-1})^k \lambda_k]^2 = 0 \qquad (2\text{-}19)$$

which since π^{-1} is non-singular is equivalent to

$$\sum_{k=1}^{N} \lambda_k^j/[(\pi^{-1})^k \lambda_k]^2 = 0 \qquad (2\text{-}20)$$

A useful implication of (2-20) is that $\lambda_k^j \neq 0$ and $\pi_j^j \neq 0$. If not since for $j \neq k$ $\lambda_k^j = -\pi_j^j/\pi_k^j$ it must be that $\pi_j^j = 0$ and so $\lambda_k^j = 0$ for every $k \neq j$. In this case since $\lambda_k^k = 1$ by definition

$$\sum_{k=1}^{N} \lambda_k^j/[(\pi^{-1})^k \lambda_k]^2 =$$

$$\lambda_k^k/[(\pi^{-1})^k \lambda_k]^2 > 0 \qquad (2\text{-}21)$$

contradicting the steady state condition (2-20).

Let H_E^N be the set of matrices π which satisfy (2-20). If H_E^N is an N^2 dimensional set then by Proposition (2-1) almost all games will have only an isolated set of points x at which $\pi(x) \epsilon H_E^N$. Define the surjective mapping $U: H_E^N \dashrightarrow H_U^N$ by

$$U_k^j = \text{sgn}\,(\pi_k^k)\,.\,\lambda_k^j/[(\pi^{-1})^k \lambda_k]^2 \qquad (2\text{-}22)$$

Since $\pi_k^k \neq 0$ this map is smooth. To show that H_E^N is $N^2 - N$ dimensional it suffices to show that H_U^N is $N^2 - N$ dimensional and diffeomorphic to H_E^N. Since U is smooth H_U^N and H_E^N are diffeomorphic provided U has a smooth inverse.

Let e be a vector of ones. By (2-20) if $U \varepsilon H_U^N$ $Ue = 0$. This shows that H_U^N is an $N^2 - N$ dimensional linear subspace.

To solve for π given U it is necessary only to solve for λ and π_j^j $j = 1,\ldots,N$. The π_k^j $j \neq k$ are then given by reversing the definition of λ in (2-10) as

$$\pi_k^j = -\pi_j^j/\lambda_k^j \qquad j \neq k \tag{2-23}$$

which is smooth since $\lambda_k^j \neq 0$. Solving for λ_k^j $j \neq k$ is straightforward since by the definition of U in (2-22)

$$U_k^j/U_k^k = \lambda_k^j \tag{2-24}$$

where $U_k^k \neq 0$ since π_k^k and λ_k^k are non-zero. To solve for π_j^j using λ and U observe from (2-23) and $\lambda_j^j = 1$ that π factors as

$$\pi = \beta d \qquad \beta_k^j = \begin{cases} 1 & j=k \\ -1/\lambda_k^j & j \neq k \end{cases} \qquad d_k^j = \begin{cases} \pi_j^j & j=k \\ 0 & j \neq k \end{cases} \tag{2-23}$$

Since $\pi_j^j \neq 0$ d is non-singular, $\pi^{-1} = d^{-1}\beta^{-1}$ and

$$(\pi^{-1})^k = (1/\pi_j^j)(\beta^{-1})^k \tag{2-26}$$

Substitute this expression into (2-22) the definition of U_j^j to find

$$U_j^j = \text{sgn}\,(\pi_j^j)\lambda_j^j/[(1/\pi_j^j)(\beta^{-1})^j\lambda_j]^2 \tag{2-27}$$

and since $\lambda_j^j = 1$ (2-27) solves as (2-28)

$$\pi_j^j = \text{sgn}\,(U_j^j)\sqrt{U_j^j}\;\left|(\beta^{-1})\,\lambda_j\right| \tag{2-28}$$

which is smooth since it can never vanish.

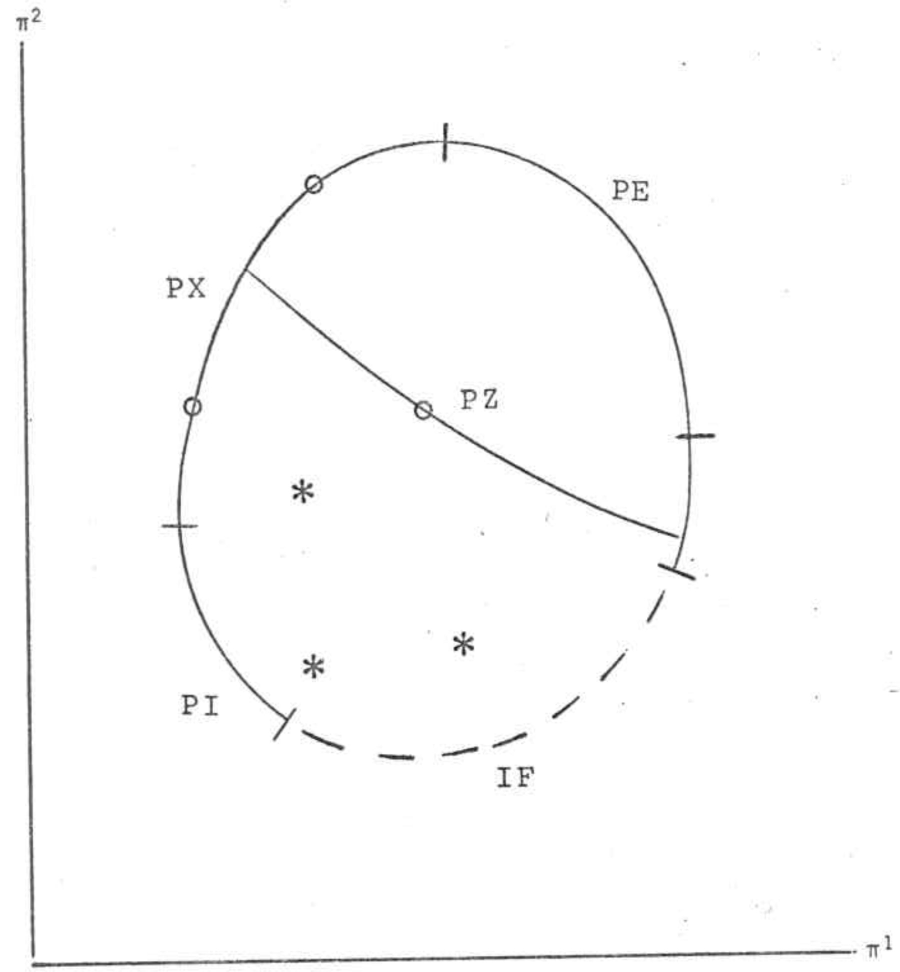
The preceding discussion can be summarized as

Proposition (2-3): In almost all games the set of steady states which are not FOPE are a set of isolated points.

Illustration of the Generic Case: Propositions (2-2) and (2-3) characterize the generic game. The FOPE form an N - 1 dimensional set which are all steady states except for an N - 2 dimensional subset. In addition there may be isolated steady states which are not FOPE.

The generic 2-agent game is illustrated in payoff space in figure (2-1). The shaded areas are all feasible payoff combinations in the game. Because the game is defined on an open set part of the boundary, labelled IF, may not be feasible. The remainder of the boundary are FOPE. Points along PE are pareto efficient. Points along PI do not pareto dominate any outcomes. Points along PX maximize $\pi^2 - \mu\pi^1$ for some $\mu > 0$. In addition to the boundary there may be interior FOPE along PZ which are saddle points of $\mu^1\pi^1 + \mu^2\pi^2$ for some $\mu \neq 0$.

Except for the isolated points labelled "0" all of the FOPE are steady states. The remaining steady states are the isolated non-autonomous steady states labelled "*". As shown the FOPE and steady states "almost" coincide.



Figure(2-1): The Generic Case

3. Stability

The previous section showed that pareto efficient points are a subset of the set of steady states. This section examines the extent to which the stable steady states coincide with the pareto efficient points.

The stability conditions in the general case are intractible and no satisfactory relationship between stability and efficiency is developed. The first half of the section limits attention to symmetric games with symmetric initial conditions. In these games steady states are stable if and only if they are (locally) pareto efficient. The second half of the section gives a sufficient condition for strong quasi-stability with non-unique steady states. As an application it is shown that symmetric pareto efficient outcomes in two firm games are strong quasi-stable even when non-symmetric perturbations are permitted.

Symmetric Games: In a symmetric game all firms are identical including in the initial conditions. Such games are easy to study since the game is characterized by two variables: the common output and the common reaction. First the steady state conditions are restated in terms of these variables. An examination of the stability conditions then shows that (local) pareto efficient points and stable steady states coincide.

Let $x^j = z$ and $R^j_k = r \quad j \neq k$ for each firm. In a symmetric game all firms are identical and these variables complete describe the game. The motion of z is given by the equations describing the motion of the x^j (1-1) and (1-6) as

$$\dot{z} = \dot{x}^j = \sum_{m=1}^{N} R_m^j \sum_{p=1}^{N} \pi_p^j R_j^p$$

$$= (\sum_{m \neq j} R_m^j + 1)(\sum_{p \neq j} \pi_p^j R_j^p + \pi_j^j)$$

$$= ((N - 1) r + 1)((N - 1) r \pi_k^j + \pi_j^j) \tag{3-1}$$

The motion of r is given by the equation of motion for R_k^j $j \neq k$ (1- 7) as

$$\dot{r} = \dot{R}_k^j = b\eta_k^j [\pi_j^k (\sum_{p=1}^{N} \pi_p^j R_k^p) + \pi_j^j y^k]$$

$$= b\eta [\pi_j^k (\sum_{p=1}^{N} \pi_p^j R_k^p) + \pi_j^j (\sum_{p=1}^{N} \pi_p^j R_j^p)]$$

$$= b\eta [\pi_k^j (\pi_j^j r + \pi_k^j + \pi_k^j (N - 2) r)$$

$$+ \pi_j^j (\pi_j^j + (N - 1) r \pi_k^j)] \tag{3-2}$$

where the motion y^k is from (1- 6) and η is the common value of the η_k^j.

Equating (3-1) and (3-2) to zero and solving for z and r shows that there are three types of steady states

$$r = 1 \qquad \pi_j^j + (N - 1) \pi_k^j = 0 \tag{3-3}$$

$$r = -1/(N - 1) \qquad \pi_j^j - \pi_k^j = 0 \tag{3-4}$$

$$r = -1/(N - 1) \qquad (N - 1) \pi_j^j - \pi_k^j - 0 \tag{3-5}$$

The steady state in (3-3) is autonomous and an extreme point of the weighted sum $\sum_{j=1}^{N} \pi^j$. The steady state in (3-4) is also autonomous and is an extreme point of the weighted sum $\pi^j - \pi^k$ $j \neq k$. It cannot be pareto efficient, since the weights do not all have the same sign. The steady state in (3-5) is not autonomous.

Necessary conditions for stability are

$$\partial\dot{z}/\partial z + \partial\dot{r}/\partial r \leq 0 \tag{3-6}$$

$$(\partial\dot{z}/\partial z)(\partial\dot{r}/\partial r) - (\partial\dot{z}/\partial r)(\partial\dot{r}/\partial z) \geq 0 \tag{3-7}$$

sufficient conditions are that (3-6) and (3-7) hold with strict inequality. Do the inefficient steady states in (3-4) and (3-5) satisfy the necessary conditions? Differentiating the expressions for $\dot{z}$ and $\dot{r}$ in (3-1) and (3-2) shows that

$$\partial\dot{z}/\partial z = ((N - 1)r + 1) \cdot$$

$$((N - 1)r\,(\pi^j_{jk} + \pi^j_{kk} + (N - 2)\pi^j_{km}) + (\pi^j_{jj} + (N - 1)\pi^j_{jk})) \tag{3-8}$$

$$\partial\dot{r}/\partial r = b\eta\,(\pi^j_k\,\pi^j_j + (\pi^j_k)^2 (N - 2) + \pi^j_j\,\pi^j_k\,(N - 1))$$

$$= b\eta\,\pi^j_k(N\,\pi^j_j + (N - 2)\,\pi^j_k) \tag{3-9}$$

when $r = -1/(N - 1)$ as in (3-4) or (3-5) we see from (3-8) that $\partial\dot{z}/\partial z = 0$, and for stability from (3-6) it must be that $\partial\dot{r}/\partial r \leq 0$. In (3-7) $\pi^j_j = \pi^j_k$ and from (3-9)

$$\partial\dot{r}/\partial r = b\eta(\pi^j_k)^2 \; 2(N - 1) > 0 \tag{3-10}$$

implying instability. In (3-5) $\pi^j_j = \pi^j_k/(N - 1)$ and from (3-9)

$$\partial\dot{r}/\partial r = b\eta(\pi^j_k)^2 \; (N/(N - 1) + N - 2) > 0 \tag{3-11}$$

implying instability as well.

The steady state in (3-3) must be broken into two cases. Define

$$S \equiv \partial^2\pi^j/\partial z^2$$

$$= \pi^j_{jj} + (N - 1)\pi^j_{jk} + (N - 1)(\pi^j_{jk} + \pi^j_{kk} + (N - 2)\;\pi^j_{km}) \tag{3-12}$$

A necessary condition for pareto efficiency is that each firm's profit be maximized subject to the symmetry constraint. The first order condition for this maximum is the condition in (3-3) $\pi^j_j + (N - 1)\pi^j_k = 0$. The second order necessary condition is $S \leq 0$. In Appendix (C) it is shown that in regular games $S < 0$ together with the first order condition are also sufficient for a symmetric outcome to be locally pareto efficient.

Examination of (3-3) shows $r = 1$ and combining this with (3-8) shows

$$\partial\dot{z}/\partial z = NS \tag{3-13}$$

while $\pi^j_j = -(N - 1)\pi^j_k$ combined with (3-9) shows

$$\partial\dot{r}/\partial r = b\eta\;(\pi^j_k)^2\;[(N - 2) - N(N - 1)]$$

$$= -\,b\eta(\pi^j_k)^2[(N - 1)^2 + 1] < 0 \tag{3-14}$$

So if $S < 0$ (3-6) holds. To determine when the other half of the sufficient condition (3-7) holds compute from (3-1), (3-2) and (3-3)

$$\partial\dot{z}/\partial r = (N - 1)((N - 1)\pi_k^j + \pi_j^j + N(N - 1)\pi_k^j)$$

$$= N(N - 1)\pi_k^j \qquad (3\text{-}15)$$

$$\partial\dot{r}/\partial z = b\eta[(\pi_j^j + \pi_k^j)S]$$

$$= - b\eta(N - 2)\pi_k^j S \qquad (3\text{-}16)$$

Using (3-13), (3-14), (3-15) and (3-16) to compute the expression in (3-7) yields

$$(\partial\dot{z}/\partial z)(\partial\dot{r}/\partial r) - (\partial\dot{z}/\partial r)(\partial\dot{r}/\partial z)$$

$$= - b\eta(\pi_k^j)^2 S[N[(N - 1)^2 + 1] - N(N - 1)(N - 2)]$$

$$= - b\eta N^2(\pi_k^j)^2 S \qquad (3\text{-}17)$$

If $S < 0$ the sufficient condition is satisfied and if $S > 0$ the necessary condition fails. Except for the unimportant case $S = 0$ a steady state is stable if and only if it is locally pareto efficient.

Strong Quasi-Stability: When steady states are not isolated it is possible to make arbitrarily small movements away from a steady state to another steady state. Stability is impossible. A weaker condition which I call strong quasi-stability requires only that a small movement away from a steady state leads to a nearby steady state. For autonomous steady states which are N - 1 dimensional this is the relevant stability concept.

Define the stability matrix

$$A = \begin{bmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial R \\ \partial\dot{R}/\partial x & \partial\dot{R}/\partial R \end{bmatrix} \quad (3\text{-}18)$$

A necessary condition for strong quasi-stability is that the real parts of the eigenvalues of A be non-positive. If steady states form an N - 1 dimensional manifold N - 1 of the N^2 eigenvalues of A must vanish, the corresponding eigenvectors indicating directions on the steady state manifold. In Levine [11] section (3) it is shown that if (x^*, R^*) is a steady state at which $N^2 - N + 1$ of the eigenvalues of A have strictly negative real parts <u>and</u> if there is an open set surrounding (x^*, R^*) in which the set of steady states are an N - 1 dimensional manifold then (x^*, R^*) and all nearly steady states are strong quasi-stable.

The previous section showed that the points x^* which are supportable as steady states are usually N - 1 dimensional. This does not tell, however, the dimensionality of steady states in (x, R) space. Appendix (D) shows that if the matrix $\lambda(x^*)$ defined in (2-10) by

$$\lambda_k^j = \begin{cases} 1 & j = k \\ -\pi_j^j/\pi_k^j & j \neq k \end{cases} \quad (2\text{-}10)$$

does not admit a non-zero vector with non-negative components in its null space, and in addition for all k $[\pi \mid \lambda_k]$ has full rank, then there is a unique R^* such that (x^*, R^*) is a steady state and a neighborhood of this steady state in which steady states form an $N - 1$ dimensional manifold.

This discussion yields the following conclusion concerning quasi-stability.

Proposition (3-1): if (x^*, R^*) is an autonomous steady state and

- o $[\pi(x^*) \mid \lambda_k]$ has full rank for all k
- o $\lambda(x^*)\mu = 0 \Rightarrow \mu \not\geq 0$
- o $A(x^*, R^*)$ has $N^2 - N + 1$ eigenvalues with strictly negative real parts

then there is an open set $U \ni (x^*, R^*)$ in which all steady states

- o form an $N - 1$ dimensional manifold
- o are strongly quasi-stable.

Two Firm Symmetric Games: Symmetric pareto efficient points in symmetric games are stable with respect to symmetric shocks. As an application of Proposition (3-1) it is demonstrated here that in two firm games symmetric pareto efficient points are strong quasi-stable; that is with respect to asymmetric as well as symmetric shocks.

The first order condition for symmetric pareto efficiency was given in (3-3) as

$$\pi_j^j + (N - 1)\ \pi_k^j = 0 \tag{3-3}$$

$$\pi_j^j + \pi_k^j = 0 \tag{3-19}$$

where (3-19) follows from N = 2. The matrix π is

$$\pi = \begin{bmatrix} -\pi_k^j & \pi_k^j \\ \pi_k^j & -\pi_k^j \end{bmatrix} \tag{3-20}$$

which has rank one since $\pi_k^j \neq 0$ and is regular singular since it admits $e = (1,1)'$ in its null space.

The matrix λ is computed from (2-10)

$$\lambda_k^j = \begin{cases} 1 & j = k \\ -\pi_j^j/\pi_k^j & j \neq k \end{cases} \tag{3-21}$$

$$\lambda = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{3-22}$$

From inspection of (3-20) and (3-22) these matrices satisfy the hypotheses of proposition (3-1). To apply the proposition requires in addition that three eigenvalues of A have strictly negative real parts.

In the two firm case the dynamic equations for agent one from (1-1) (1- 6) and (1- 7) are

$$\dot{x}^1 = [\pi_1^1 + R_1^2\pi_2^1 + R_2^1(\pi_2^2 + R_2^1\pi_1^2)] \tag{3-23}$$

$$\dot{R}_2^1 = b\eta[\pi_1^2\,(\pi_1^1R_2^1 + \pi_2^1) + \pi_1^1(\pi_2^2 + \pi_1^2R_2^1)] \tag{3-24}$$

Differentiating these equations, using symmetry and the equilibrium conditions in (3-19) shows that

$$\partial\dot{x}^1/\partial R_2^1 = [\pi_2^2 + R_2^1\;\pi_1^2 + R_2^1\;\pi_1^2] = b\pi_k^j$$

$$\partial \dot{x}^1 / \partial R_1^2 = \pi_k^j$$

$$\partial \dot{R}_2^1 / \partial R_2^1 = - b\eta \, (\pi_k^j)^2$$

$$\partial \dot{R}_2^1 / \partial R_1^2 = 0 \qquad (3\text{-}25)$$

By symmetry the stability matrix A defined in (3-18) is

$$A = \begin{bmatrix} \partial \dot{x}^1/\partial x^1 & \partial \dot{x}^1/\partial x^2 & \pi_k^j & \pi_k^j \\ \partial \dot{x}^1/\partial x^2 & \partial \dot{x}^1/\partial x^1 & \pi_k^j & \pi_k^j \\ \partial \dot{R}_2^1/\partial x^1 & \partial \dot{R}_2^1/\partial x^2 & - b\eta(\pi_k^j)^2 & 0 \\ \partial \dot{R}_2^1/\partial x^2 & \partial \dot{R}_2^1/\partial x^1 & 0 & - b\eta(\pi_k^j)^2 \end{bmatrix} \qquad (3\text{-}26)$$

Let $e_0 = (0, 0, 1, -1)'$; then $Ae_0 = -b\eta(\pi_k^j)^2 e_0$ so e_0 is an eigenvector corresponding to a negative eigenvalue. Furthermore there are two other eigenvalues corresponding to symmetric departures from equilibrium and by the analysis of symmetric shocks these are strictly negative as well.

Proposition (3-1) now applies and the efficient steady state and nearby steady states are strong quasi-stable.

4. The Very Long Run

The system we have been studying was derived by approximating the true equation for y (1- 5) by (1- 6). Thus the true system is a (small) perturbation of the system we have studied. How does this perturbation affect our results? A good reference for the discussion that follows is Hirsch and Smale [8] chapter 16.

The first point is that a small perturbation can't have steady states far distant from the steady states of the unperturbed system. This means that the perturbed steady states are (approximately) a subset of the steady states we have studied. In effect, a perturbation can destroy steady states and it can introduce new steady states but the new ones must be close to the old ones.

A steady state is hyperbolic iff none of the real parts of the eigenvalues of the stability matrix vanish there. In a neighborhood of a hyperbolic steady state a perturbation creates no new steady states, it shifts the hyperbolic steady state only a small distance, and it preserves the stability properties of the hyperbolic steady state. In the symmetric case the perturbation is symmetric and the steady states are hyperbolic (generically) with respect to symmetric states. Thus small perturbations have no important effect on the results given in the symmetric case. Similarly, exceptional steady states are generically hyperbolic and thus won't be affected much by small perturbations.

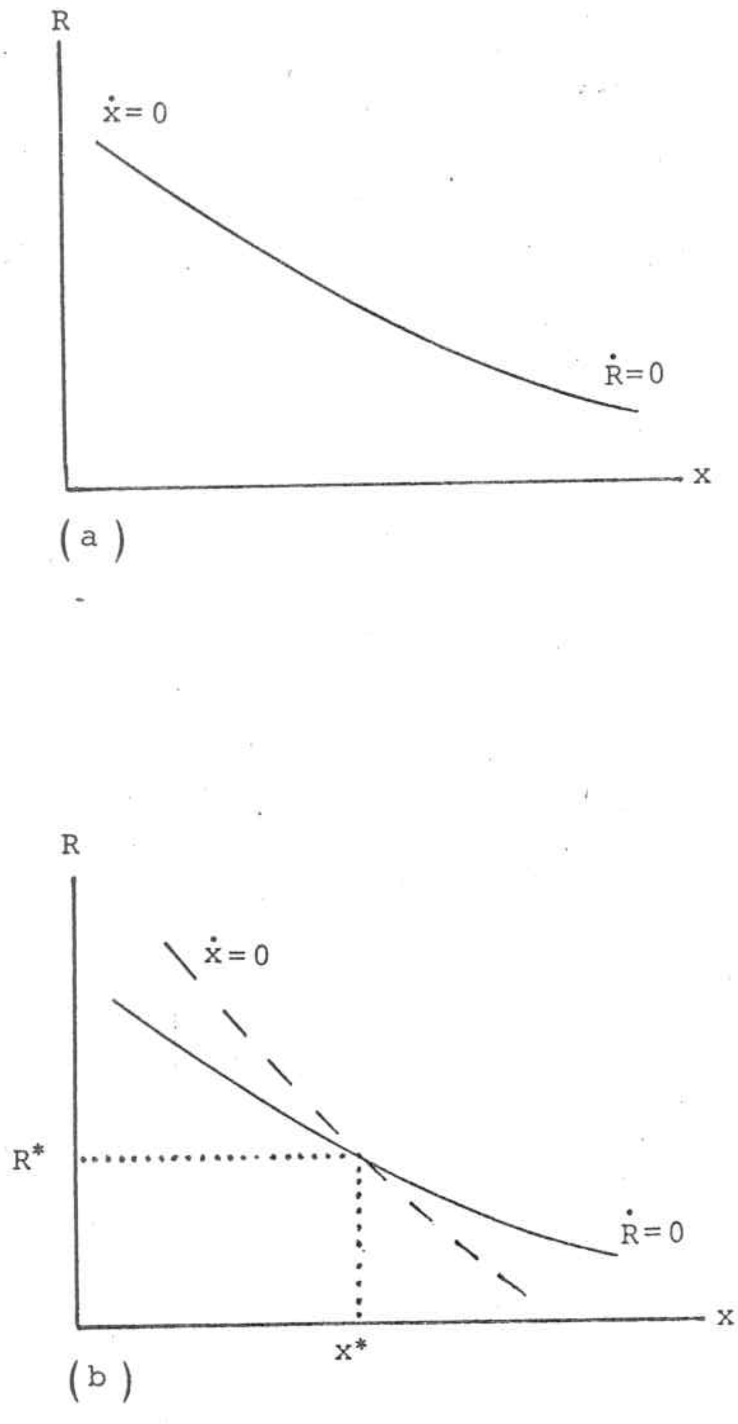
The situation near the manifold of autonomous steady states is very different. Generically, perturbations of a dynamical system have locally isolated steady states. It is to be expected that when the system is perturbed the steady state manifold will be replaced by a set of isolated steady states lying near the manifold.

This deserves a bit of explanation. Figure (4-1) illustrates what the system might look like before and after a perturbation. Initially the $\dot{x} = 0$ and $\dot{R} = 0$ curves coincide constituting a manifold of steady states. When (1- 6) is replaced by (1- 5) the $\dot{x} = 0$ curve shifts slightly by an amount proportional to b. Only the steady state (x*, R*) remains.

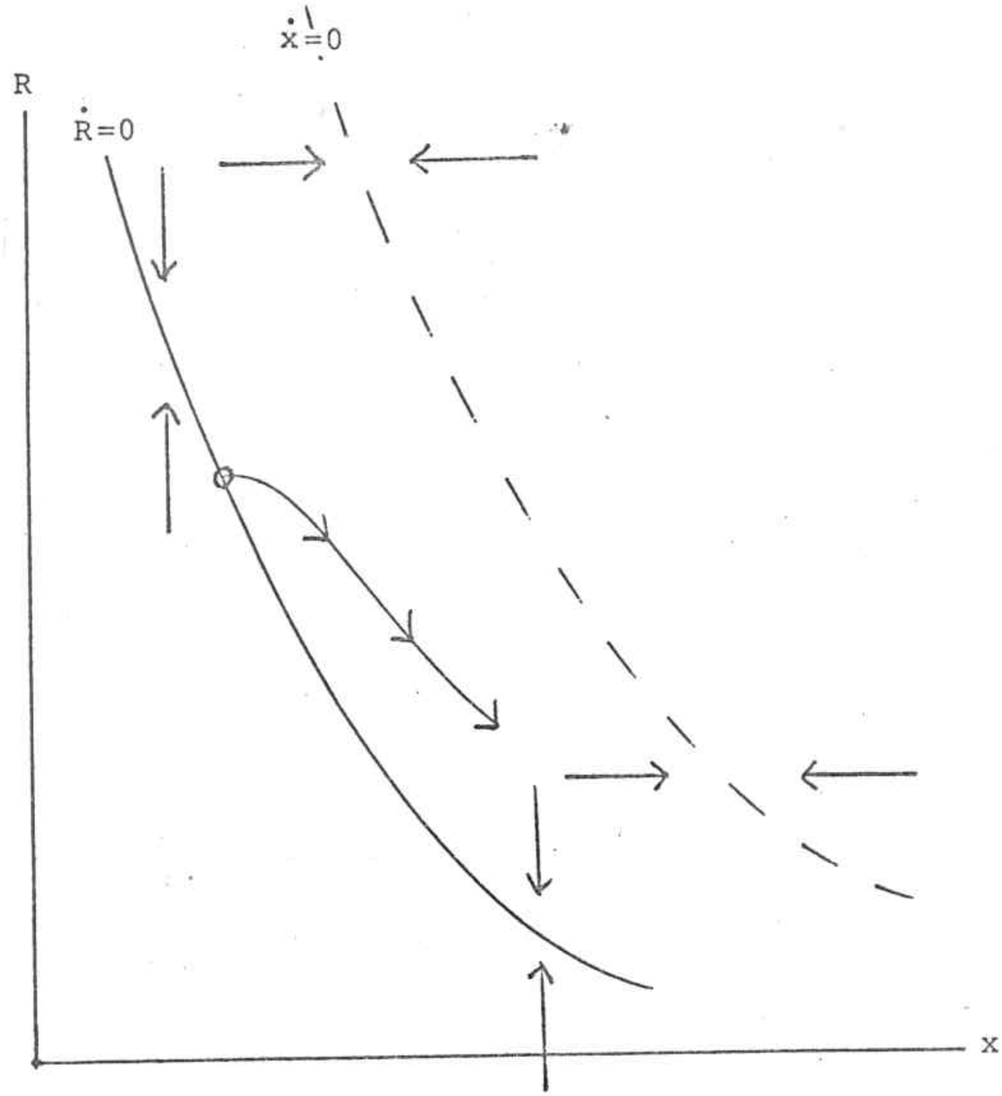
What happens along the old steady state manifold? Suppose the old manifold was stronge quasi-stable so both $\dot{x} = o$ and $\dot{R} = 0$ are attractors. Initially R doesn't move, while x moves towards the new $\dot{x} = 0$ curve. As x approaches $\dot{x} = 0$ R is no longer in equilibrium and begins to move along the old steady state manifold as illustrated in Figure (4-2).

If the old steady state manifold is unstable the situation is quite different: a small perturbation of the system will typically cause the system to move away from the old manifold entirely.

In the strong quasi-stable case, how rapid is the drift along the steady state manifold? From (1- 5) and (1- 7) $\dot{x} = bF(x, R)$ and $\dot{R} = b^2 G(x, R)$ where remember b is small. As the system moves along the old manifold x is approximately in equilibrium at $\dot{x} = 0$ since it equilibrates faster than $\dot{R}$. This means, since $\dot{x} = 0$ and $\dot{R} = 0$ lie apart by order b, that the distance of the system from $\dot{R} = 0$ is $\Delta x \cong bH(R)$. Thus, $\dot{R} = b^2 G_x \Delta x = b^3 G_x H(R)$ where G_x are the derivatives of G with respect to x.



Figure(4-1): Perturbation of the Steady State Manifold



Figure(4-2): Drift Along the Unperturbed Steady State Manifold

In the short run x equilibrates most rapidly at rate $O(b)$ so R is determined by initial conditions and x by $\dot{x} = 0$. In the long run R equilibrates at rate $O(b^2)$ causing the system to move towards the unperturbed manifold of steady states. In the very long run, however, the perturbation causes the system to drift along the unperturbed steady state manifold at rate $O(b^3)$. This is similar to the notion of fast and slow manifolds introduced by Zeeman [13] chapter 3.

The exact nature of very long run steady states where output shares are determinate is an interesting question for future research.

5. Conclusion

When firms can engage in costless retaliatory policies we anticipate that they will reward opponents who make pareto improving output adjustments and punish those who selfishly try to increase output. This paper has shown that, subject to technical qualifications, this is true. The important qualifications:

- In asymmetric games pareto efficient outcomes may be unstable steady states. A small subset of the pareto efficient outcomes may not be steady states at all.
- In asymmetric games there may be stable steady states which satisfy the first order conditions for pareto efficiency, but not the second order conditions. A small number of steady states may not even satisfy the first order conditions.

APPENDIX (A)--Singularity and Regular Singularity

The objective is to prove two lemmas:

Lemma (A-1): $m \times n$ matrices of rank r are a set of codimension $(m - r) \times (n - r)$.

Lemma (A-2): regular singular $n \times n$ matrices are a set of codimension 2.

Proof of (A-1): From McCoy [14] section 15.5 a matrix A of rank r has an $r \times r$ non-singular submatrix A_{11}. Assume

$$A = \begin{array}{c} \begin{array}{cc} r & n-r \end{array} \\ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{array} \begin{array}{l} r \\ m-r \end{array} \qquad \text{(A-1)}$$

Any matrix of rank r is obtained from a matrix of the form (A-1) by permuting rows and columns. Since only a finite set of such permutations is possible it suffices to prove the lemma for a matrix of the form (A-1). Following Guilleman and Pollack [6] chapter 1.4 problem 13 define

$$B = \begin{bmatrix} I & -A_{11}^{-1} A_{12} \\ 0 & I \end{bmatrix} \qquad \text{(A-2)}$$

Since B is nonsingular rank (AB) = rank (A). Also

$$AB = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} - A_{21}A_{11}^{-1} A_{12} \end{bmatrix} \tag{A-3}$$

so that A has rank r if and only if

$$F(A) = A_{22} - A_{21} A_{11}^{-1} A_{12} = 0 \tag{A-4}$$

Since $\partial F/\partial A_{22} = I$ the transformation F has full rank. Thus since F(A) is (m - r) x (n - r) the lemma follows. Q.E.D.

Proof of (A-2): Let A be square, singular and non-regular. Any such matrix can be obtained from a matrix that admits a vector of the form $(0, X_2, \ldots, X_n)'$ in its null space by a finite permutation of columns, so assume A has this form. Set

$$A = [a_{11}, A_{12}] \tag{A-5}$$

where a_{11} is an n-vector. Then A has the required form if and only if A_{12} has rank (n - 2) or less. By the same reasoning used in lemma (A-1) this implies a codimension of 2. Q.E.D.

APPENDIX (B)--Genericity in Regular Games

Let G be the set of C^2 mappings $\pi^{\cdot}: \mathcal{X} \dashrightarrow \mathbb{R}^N$ in the Whitney C^2 topology where $\mathcal{X}$ is open in $\mathbb{R}^N$. Let G^R be the subset of $\pi^{\cdot} \varepsilon$ G which also satisfy $\pi_k^j(x) < 0 \quad j \neq k$ and det $(\pi(x)) \neq 0$. The objective is to prove

Lemma (B-1): D residual in G implies D dense in G^R.

From general topological considerations it suffices to prove that $G^R \subset$ closure (interior G^R)). Define G^1 to be the subset of G which satisfies $\pi_k^j(x) < 0 \quad j \neq k$ and G^2 the subset which satisfies det $(\pi(x)) \neq 0$. Obviously small perturbations of π satisfying $\pi_k^j(x) < 0$ will satisfy the restriction so G^1 is open. Thus it suffices to show $G^2 \subset$ closure (interior G^2)). Define G^3 to be the subset of G^2 such that for $\pi^{\cdot} \varepsilon G^3$ there is an x_1 and x_2 with det $(\pi(x_1)) > 0$ and det $(\pi(x_2)) < 0$. Obviously G^3 is open. To prove lemma (B-1) it is then necessary to only show that

Lemma (B-2): $G^2 \subset$ closure (G^3)

This requires a preliminary lemma.

Lemma (B-3): If A is singular there is a matrix B such that for $\lambda > 0$ det $(A + \lambda B) > 0$.

Proof:

By the Jordan decomposition theorem $A = C^{-1}JC$ where J is upper triangular, has zeroes in the first $(N - r)$ diagonal positions and the non-zero eigenvalues of A in the remaining r diagonal positions. Assume without loss of generality that the product of the non-zero eigenvalues of A is positive. Let D be a diagonal matrix with ones in the first r positions, zeroes in the other $(N - r)$. Then for $\lambda > 0$ $\det(J + \lambda D) > 0$. Thus $B = C^{-1}DC$ satisfies the required property.

Q.E.D.

Now suppose $\pi^{\cdot} \in G^2 - G^3$. Ignoring the case $\det(\pi)$ vanishes identically (left as an exercise) suppose without loss of generality for some x_1 $\det(\pi(x_1)) < 0$. By assumption for some x_2 $\det(\pi(x_2)) = 0$. Let B be a ball centered on x_2. Because $\mathcal{X}$ is open in $\mathbb{R}^N$ we may assume $B \subset \mathcal{X}$ and $x_1 \notin$ closure (B). Let π^* be (by lemma B-3)) such that $\det(\pi(x_2) + \lambda\pi^*) > 0$ for $\lambda > 0$. Using techniques similar to those of chapter 2 section 2 of Hirsch [7] there is a function $\bar{\pi}^{\cdot}: \mathcal{X} \to \mathbb{R}^N$ which is C^{∞}, vanishes outside closure (B) and has $D\bar{\pi}^{\cdot}(x_2) = \pi^*$. Then $(\pi^{\cdot} + \lambda\,\bar{\pi}^{\cdot}) \in G^3$ for $\lambda > 0$ and approximates π^* arbitrarily well. This proves lemma (B-2) and thus (B-1).

APPENDIX (C)--Symmetric Efficiency

The objective is to show that in a symmetric game with $\pi_k^j < 0 \; j \neq k$ $\pi_j^j + (N - 1)\, \pi_k^j = 0$ and $S < 0$ imply local efficiency. It was already shown that this implies no small symmetric perturbation makes any firm better off. Suppose π has rows π^j as always. I will show that if z is an asymmetric N-vector then $\pi_k z < 0$ for some k thus implying any non-symmetric perturbation makes at least one firm worse off via the mean value theorem.

<u>Lemma (C-1)</u>: If z is non-symmetric for some k $\pi_k z < 0$.

<u>Proof</u>: Suppose conversely z is asymmetric and $\pi z \geq 0$. Since $\pi_k^j \neq 0$ $j \neq k$ a check shows that π has rank N - 1. Since $\pi e = 0$ where e is symmetric (by assumption) it cannot be that $\pi z = 0$. Thus we may assume $\pi z \geq C$ where $C_1 = 1 \;\; C_k = 0 \;\; k > 1$. By a theorem on linear inequalities found for example as theorem 2.7 in Gale [3] the system $\pi z \geq C$ has a solution if and only if the system $y'\pi = 0 \;\; y'C = 1$ has no non-negative solution. Letting $e = (1, \ldots, 1)'$ we see that $e'\pi = 0$ and $e'C = 1$. Thus $\pi z \geq C$ has no solution, a contradiction.

Q.E.D.

APPENDIX (D)--The Manifold of Autonomous Steady States

The objective is to prove that when the hypotheses of proposition (3-1) are satisfied at (x^*, R^*) there is an open set $U \supset (x^*, R^*)$ in which steady states forma manifold of dimension (N - 1). This is done by means of four lemmas.

Lemma (D-1): If $\lambda(x^*)$ does not admit a weak positive null vector then there is an open $U \supset x^*$ such that $x \in U$ and (x, R) a steady state implies $\pi(x)$ is singular.

Proof: Since $\lambda(x)$ is continuous choose U so that $x \in U$ implies $\lambda(x)$ has no weak positive null vector. If $\pi(x)$ is non-singular the steady state is exceptional and from (2-20) satisfies $\sum_{k=1}^{N} \lambda_k^j / [(\pi^{-1})^k \lambda_k]^2 = 0$ contradicting the fact λ doesn't admit weak positive null vectors.

Q.E.D.

Lemma (D-2): If $[\pi(x^*) \mid \lambda_k(x^*)]$ has full rank for all k then there is open $U \supset x^*$ such that $x \in U$ and $\pi(x)$ singular imply (x, R) can't be a non-autonomous steady state.

Proof: Since $\pi(x)$ and $\lambda(x)$ are continuous choose U so that $x \in U$ implies $[\pi(x) \mid \lambda_k(x)]$ has full rank for all k. From (2-14) a necessary condition for (x, R) to be a steady state is

$$[\pi \mid -\lambda_k] \begin{bmatrix} R_k \\ y^k \end{bmatrix} = 0 \qquad \text{all } k \tag{D-1}$$

Since π is singular and $[\pi \mid \lambda_k]$ has full rank the only solution to (D-1) is $\pi R_k = 0$ and $y^k = 0$. Thus all $y^k = 0$ and if (x, R) is a steady state it is an autonomous one. Q.E.D.

Corollary D-3: If (x*, R*) are as in proposition (3-1) there is $U \supset (x^*, R^*)$ in which all steady states are autonomous.

Now define $w_k^j = \pi^j R_k$. From (1-1) and (1- 7) the equations of motion are

$$\dot{x}^j = \sum_{k=1}^{N} R_k^j w_k^k$$

$$\dot{R}_k^j = b\eta_k^j \, [\pi_j^k w_k^j + \pi_j^j w_k^k] \tag{D-2}$$

Let $\pi_{-k}^j = (\pi_1^j, \ldots, \pi_{k-1}^j, \pi_{k+1}^j, \ldots, \pi_N^j)$. The derivative of w_k^j with respect to all the state variables is the row vector

$$Dw_k^j = [\underset{1}{(D\pi^j R_k)'} \mid 0 \mid \ldots \mid \underset{k}{\pi_{-k}^j} \mid \ldots \mid 0] \tag{D-3}$$

where $D\pi^j$ is the matrix of second derivatives of π^j. Observe that $R_k^* = \gamma/\gamma_k$ so that this can be written as

$$Dw_k^j = [(D\pi^j \gamma)'/\gamma_k \mid 0 \mid \ldots \mid \pi_{-k}^j \mid \ldots \mid 0] \tag{D-4}$$

The matrix Dw is formed by stacking the Dw_k^j.

Lemma (D-4): If U contains only autonomous steady states and at steady states corank (Dw) $\leq$ (N-1) then autonomous steady states are a manifold in U.

Proof: Since π is continuous by choosing U small enough we may assume whenever π is singular the first row (say) is a linear combination of the other rows. Since all steady state in U are autonomous w = 0 is necessary and sufficient for a steady state there. Define $\bar{w}$ to be the sub-matrix of w formed by deleting the (N-1) rows corresponding to $w_2^1, w_3^1, \ldots, w_N^1$. I claim that $\bar{w} = 0$ implies w = 0 in U and $D\bar{w}$ has full rank, which will prove the lemma.

Suppose $\bar{w} = 0$. Then $w_1^1, w_1^2, \ldots, w_1^N = 0$ which reads $\pi R_1 = 0$. Since $R_1^1 = 1$ this implies π is singular. Does $w_k^1 = 0$? This reads $\pi' R_k = 0$. But $\pi^j R_k = 0$ for $j \neq k$ and π^1 is a linear combination of the π^j. Thus $\pi' R_k = 0$. So w = 0.

Since corank (Dw) $\leq$ (N-1) to show $D\bar{w}$ has full rank it suffices to show that Dw_k^1 is a linear combination of the rows of $D\bar{w}$. Observe from (D-4) that

$$\gamma_k Dw_k^m - \gamma_1 Dw_1^m = [-\gamma_1 \pi_{-1}^m \mid 0 \mid \ldots \mid \gamma_k \pi_{-k}^m \mid 0] \tag{D-5}$$

Thus

$$\gamma_k Dw_k^1 - \gamma_1 Dw_1^1 = [-\gamma_1 \pi_{-1}^1 \mid 0 \mid \ldots \mid \underset{k}{\gamma_k \pi_{-k}^1} \mid 0] \tag{D-6}$$

Suppose $\pi^1 = \sum_{i=2}^{N} \mu_1 \pi^i$. Then from (D-5) and (D-6)

$$Dw_k^1 = (1/\gamma_k)[\gamma_1 Dw_1^1 + \sum_{i=2}^{N} (\gamma_k Dw_k^1 - \gamma_1 Dw_1^i)\mu_i] \qquad \text{(D-7)}$$

is the desired linear combination. Q.E.D.

Lemma (D-5): At an autonomous steady state if the stability matrix has corank (A) $\leq$ (N-1) then corank (Dw) $\leq$ (N-1).

Proof: From (D-2) the equations of motion can be written as

$$\dot{v} = Lw \qquad \text{(D-8)}$$

where v is the vector of state variables and L is an $N^2 x N^2$ matrix. Since w = 0 at the autonomous steady state A = LDw. Thus corank (A) $\leq$ (N-1) implies corank (Dw) $\leq$ (N-1). Q.E.D.

The proof of proposition (3-1) now follows from the fact that corollary (D-3) and lemma (D-5) imply the hypothesis of lemma (D-4).

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